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Le GRADE de DOCTEUR D'ETAT
ES-SCIENCES PHYSIQUES

PAR

Bernadette DORIZZI

SUJET : ETUDE DE L'INTEGRABILITE DES SYSTEMES DYNAMIQUES

soutenue le 15 Décembre 1983 devant la Commission d'examen

MM. R. TEMAM Président
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ABSTRACT :

In this thesis, we treat the integrability of dynamical systems modelised by nonlinear ordinary-differential equations.

The singularity analysis of the solutions of the equations of motion furnishes a very useful criterion for the selection of potentialy integrable systems (Painlevé property). Several new integrable systems, Hamiltonian and dissipative are found, related to this property.

During this study, we have been led to weaken the original Painlevé property, in order to deal with a more important number of integrable systems.

Then, the same questions of chaos and integrability are asked for quantum systems : We have studied, in particular, the quantification of classicaly integrable systems and a simple quantum model, the classical analogous of which, is known to possess a chaotic behaviour.

REMERCIEMENTS

Je remercie Messieurs J.P. Poitevin, F. Ducastel, J.M. Chaduc, H. Zimmermann et M. Minoux pour m'avoir permis, au sein du Département M.T.I. du CNET, de mener à bien l'élaboration de cette thèse. Je suis tout particulièrement reconnaissante à R. Padjen qui m'a accueillie dans le groupe et m'a toujours témoigné une grande confiance.

Je remercie Monsieur le Professeur R. Temam pour avoir accepté de présider le jury de cette thèse et Monsieur le Professeur C. Bardos pour y avoir participé.

Monsieur le Professeur R. Caboz, avec sa gentillesse habituelle m'a aidé, par ses remarques judicieuses, à améliorer la version définitive de cette thèse. Qu'il en soit remercié.

Y. Pomeau a, tout au long de ces années de travail, été présent au sein de notre équipe. Je voudrais le remercier pour ses nombreux conseils et suggestions ainsi que pour la confiance qu'il a faite à la "jeune physicienne" que je suis, en acceptant de diriger cette thèse.

La plupart de ces travaux ont été fait en collaboration avec A. Ramani et B. Grammaticos. Il n'est pas de mots pour les remercier. Le climat d'amitié, de détente et de création "folle" qu'ils ont su créer, allié à leur dynamisme et enthousiasme naturel a été très important pour moi. A leur contact, j'ai peu à peu appris les rudiments du métier de physicien, l'art et la manière de poser un calcul ou de réfléchir à partir d'un résultat.

Autour de notre petit groupe de trois, d'autres amis ont aussi évolué.

Merci à V. Papageorgiou qui, avec sa présence discrète mais très efficace au sein du groupe, m'a aidé entre autre à m'initier aux techniques délicates des groupes de Lie. Je voudrais aussi remercier T. Bountis pour la collaboration que nous avons eu lors de son séjour en Europe.

Cette thèse a nécessité, au moins en partie, des calculs numériques, et je remercie C. Sébastien pour son aide.

J'ai beaucoup apprécié les collaborations que j'ai eues avec J. Hietarinta et P. Winternitz. Qu'ils soient remerciés pour l'ouverture qu'ils ont pu donner à mon travail.

A l'occasion de conférences à l'étranger, j'ai eu le plaisir de pouvoir rencontrer plusieurs physiciens . En particulier, j'ai pu avoir des discussions enrichissantes avec M. Ablowitz, H. Ségur et A. Fokas.

C'est lors de mon récent voyage à Princeton que j'ai fait la connaissance de M.D Kruskal. Il est difficile d'exprimer ce que la rencontre d'un homme d'une personalité aussi riche et aussi variée peut apporter mais je voudrais au moins le remercier pour nos nombreuses discussions qui ont fortement influencé le déroulement de mes travaux.

B. DORIZZI

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CHAPITRE 1

INTRODUCTION

L'étude des systèmes dynamiques a connu un développement spectaculaire ces dernières années. Les théories élaborées pour ces systèmes intéressent des branches très variées des sciences appliquées. Malgré cette grande diversité, des caractères universels peuvent être dégagés de la notion de système dynamique.

En général, on s'intéresse à l'étude de l'évolution, c'est-à-dire du comportement en fonction du temps des systèmes physiques. Le caractère commun de tous ces systèmes est leur aspect déterministe : il suffit de connaître l'état du système à un instant donné (ou son histoire jusqu'à cet instant) pour déterminer cet état à l'instant suivant. On peut rendre cette notion plus concrète par l'intermédiaire d'un exemple emprunté à la mécanique hamiltonienne.

Considérons un ensemble de particules en interaction dont l'hamiltonien est donné par $H = H(x,p)$.

Si on connaît (avec une précision suffisante) l'état du système à l'instant t_0, état qui est ici décrit par le vecteur $[x_i(t_0), p_i(t_0)]$, on peut calculer l'état du système à un instant ultérieur t_1 : $[(x_i(t_1), p_i(t_1)]$. Il suffit pour cela d'intégrer les équations qui modélisent le système : les équations de Hamilton.

Ce caractère déterministe est présent non seulement dans le cadre d'une description classique du système mais aussi dans une description quantique bien que, pour cette dernière, on ne sache définir qu'une densité de probabilité.

Le rôle du vecteur d'état est joué maintenant par la fonction d'onde qui sert à calculer cette densité de probabilité. Or, l'évolution de cette fonction d'onde est obtenue de façon déterministe, via l'équation de Schrödinger.

Comme on l'a vu dans l'exemple ci-dessus, la modélisation des systèmes dynamiques procède par l'intermédiaire d'un système d'équations (ce qui rend particulièrement commode leur étude à l'aide des simulations numériques sur ordinateur). De façon concrète, on a le plus souvent, une des formes suivantes :

□ *Des applications*, comme par exemple "le mapping logistique"

$$x_{n+1} = 4\,a x_n(1-x_n) \qquad (0 < x_n \leqslant 1)$$

ou à deux dimensions, l'application de Hénon

$$\begin{cases} x_{n+1} = 1 - cx_n^2 + y_n \\ y_{n+1} = \beta x_n \end{cases}$$

□ *Des équations différentielles ordinaires* (O.D.E.) *ou des systèmes*

A titre d'exemple, citons l'équation de Van der Pol qui a été introduite dans l'étude des circuits électriques :

$$\ddot{y} = \varepsilon\dot{y}(1-y^2) - ay \qquad (\varepsilon > 0,\ a > 0)$$

et le système que Lorenz a proposé pour modéliser l'instabilité de Bénard en hydrodynamique :

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = -xz + rx - y \\ \dot{z} = \ \ xy - bz \end{cases}$$

□ *Des équations aux dérivées partielles* (P.D.E.)

Citons, parmi la multitude d'exemples possibles, l'équation de Korteweg de Vries (K.D.V.)

$$u_t - 6uu_x + u_{xxx} = 0$$

qui fut proposée pour la propagation des ondes de surface, et le système d'équations introduit par Hirota-Satsuma

$$\begin{cases} u_{xxx} + 6uu_x - 2u_t + 4b\varphi\varphi_x = 0 \\ \varphi_{xxx} + 3u\varphi_x + \varphi_t = 0 \end{cases}$$

qui décrit l'interaction de deux ondes longues de relations de dispersion différentes.

□ *Des équations intégrales* qui sont nécessaires lorsque le futur d'un système dépend de tous les états antérieurs.

Volterra a ainsi suggéré l'équation :

$$\frac{1}{y}\frac{dy}{dt} = a + by + \int_c^t K(t,s)\ y(s)\ ds$$

pour modéliser la croissance d'une population y, où la croissance est influencée par un facteur proportionnel à la population, par une influence inhibitrice proportionnelle au carré de la population et par une composante héréditaire composée de la somme de facteurs individuels rencontrés dans le passé.

Dans toute cette thèse, nous allons nous limiter à l'étude de systèmes modélisés par des équations différentielles.

Une classe de telles équations a été étudiée en détail depuis fort longtemps. Il s'agit des équations linéaires.

La structure des solutions est simple : la solution générale s'obtient comme combinaison linéaire d'un nombre fini de solutions particulières indépendantes.

De telles équations sont considérées comme intégrables car même si dans la plupart des cas on ne sait pas exprimer les solutions sous forme fermée, on peut au moins avoir une idée de leur comportement local à l'aide des coefficients de l'équation.

Un grand nombre d'équations linéaires du second ordre ont ainsi servi à définir de nouvelles fonctions : fonctions hypergéométriques, de Bessel, de Legendre, etc.

Le cas non linéaire est beaucoup plus compliqué. Si, en général, il n'existe pas de principe de superposition pour de telles équations, certaines familles, comme les équations de Riccati, possèdent encore cette propriété. Il se trouve que, de plus, les singularités de ces équations (dans le plan complexe du temps) ne sont pas trop "mauvaises". Fuchs et Liouville ont en effet montré, au début du siècle, que leurs seules singularités mobiles (dépendantes des conditions initiales) sont des pôles. Dans le même esprit, Painlevé a classifié les équations du second ordre à singularités critiques (singularités essentielles ou coupures algébriques) fixes.

Il n'y a pas eu de développements importants dans le domaine du non linéaire jusqu'à ces dernières années, où un regain d'intérêt, dû à certaines découvertes récentes, s'est fait sentir.

Il s'agit tout d'abord d'un certain nombre de travaux concernant les ondes solitaires et l'équation de Korteweg de Vries. Suivant la voie ouverte par Fermi, Pasta, Ulam, en 1955, Zabusky et Kruskal ont étudié l'équation de Korteweg de Vries et ils ont découvert que, lorsque deux ondes solitaires de l'équation de Korteweg de Vries rentrent en collision, elles le font de manière élastique et conservent leur forme initiale.

Cette remarque a permis d'importants développements dans le domaine des p.d.e non linéaires.

Une autre découverte importante a été l'observation de phénomènes chaotiques au cours de l'évolution de certains systèmes dynamiques d'abord par Rikitake, puis un peu plus tard par Lorenz (1963).

Ce dernier, s'intéressant aux équations de Navier-Stokes en hydrodynamique, a étudié une réduction à un système de trois équations couplées, qui porte son nom. Il a pu mettre en évidence, numériquement et par des arguments analytiques, que la solution de ces équations restait dans une partie de l'espace des phases de structure géométrique très particulière.

Dans le même esprit Ruelle et Takens en 1971, ont pu mettre en évidence pour le système de Lorenz, un mécanisme conduisant au chaos, introduisant ainsi pour la première fois le terme "d'attracteur étrange".

L'extension au système initial des équations de Navier-Stokes est, pour le moment, encore mal connue.

De manière générale, on peut d'ailleurs souligner que des notions comme celles de chaos, d'attracteur, ont été introduites dans le cadre des o.d.e.

L'extension aux p.d.e est encore un problème ouvert que nous n'allons pas du tout aborder dans ce travail.

Lorsque l'on étudie un système dynamique, on est préoccupé par le comportement à long terme du système. Si ce long terme ne se réalise pas toujours physiquement à l'échelle humaine (dans le cas du système planétaire, le temps caractéristique peut être de l'ordre du milliard d'années), pour bon nombre de systèmes physiques c'est un état que l'on peut observer en pratique (par exemple, pour des particules dans les anneaux de stockage d'un accélérateur, l'évolution est assez longue déjà au bout de quelques minutes).

Une étude assez grossière peut déjà montrer un changement qualitatif du comportement du système pour des temps longs.

Ainsi, pour certains systèmes, on observe une perte de stabilité ; les trajectoires partent à l'infini. Lorsque le mouvement reste borné, on espère pouvoir obtenir des informations supplémentaires sur la localisation des trajectoires dans l'espace des phases. Si l'on étudie, par exemple, le mouvement de l'oscillateur harmonique à deux dimensions, on observe que les différentes trajectoires s'enroulent sur un tore. Mais cette situation simple n'est pas la plus générale. Dans le cas d'un système hamiltonien borné de dimension N, certaines trajectoires peuvent remplir en totalité ou en partie les variétés de dimension 2N-1 d'énergie constante. Par contre, lorsque le système contracte les volumes, des attracteurs de diverses natures peuvent apparaître, soit classiques (points fixes ou cycles limites), seuls présents en dimension plus petite ou égale à deux, soit de nature chaotique, appelés "étranges".

Dans les cas hamiltoniens aussi bien que dissipatifs, on parle alors de comportement chaotique. Celui-ci est associé à une grande irrégularité combinée à une dépendance sensible aux conditions initiales, c'est-à-dire que deux conditions initiales arbitrairement proches peuvent donner lieu à des trajectoires arbitrairement éloignées. Pour un tel système, il est donc impossible de prédire, même numériquement, le comportement à long terme à partir des conditions initiales.

Si l'on s'intéresse à la prédictibilité à long terme, il peut être intéressant d'avoir certains critères qui permettent d'affirmer qu'un système n'est pas chaotique.

Nous allons illustrer ceci à l'aide du système que Rikitake a proposé pour la description de la variation du champ magnétique terrestre.

Il s'écrit :

$$\begin{cases} \dot{x} = -\mu x + \beta y + yz \\ \dot{y} = -\mu y - \beta x + xz \\ \dot{z} = -xy + \alpha \end{cases}$$

Le comportement générique de ce système est chaotique.

Cependant, lorsque $\alpha = 0$ et $\mu = 0$, il existe deux constantes du mouvement indépendantes du temps et l'on peut écrire la solution du système au moyen de quadratures. En conséquence, à partir d'une condition initiale donnée (x_0, y_0, z_0), on peut prédire la solution $[x(t), y(t), z(t)]$ à un instant t arbitraire et donc avoir ainsi une description exacte des trajectoires du système.

Plus généralement, lorsque la résolution d'un système différentiel se ramène à des quadratures, ce système est intégrable et donne lieu à un comportement non chaotique.

Mais, qu'entend-t-on communément par système intégrable ?

- Tout d'abord un système où l'on peut écrire la solution explicitement. Dans certains cas, l'écriture explicite de la solution nécessite la résolution d'un problème algébrique. C'est ce qui se produit, par exemple, lorsqu'un système formé de N O.D.E possède N constantes du mouvement, dépendantes du temps.

- Un système où les équations du mouvement se ramènent à des quadratures.

- Pour un système hamiltonien de rang N, la définition la plus générale est celle de Liouville : Le système est intégrable s'il existe N intégrales globales , indépendantes et en involution. On peut alors calculer N autres intégrales, *locales*, et ainsi trouver un système de coordonnées locales dans lequel les équations de mouvement se séparent.

- Ainsi qu'il en a été fait mention précédemment, la classe des équations linéaires peut être considérée comme intégrable. Ceci est vrai même si les coefficients de l'équation ne sont pas constants et que l'on ne sait pas écrire les solutions explicitement.

- Par extension, considérons l'équation non linéaire suivante (équation de Riccati) :

  $$\frac{dy}{dt} = a + by + cy^2$$

  où a, b, c sont des fonctions de t.

  Si on pose :

  $$y = \frac{u'}{cu}$$

  u vérifie une équation linéaire du deuxième ordre, homogène, à coefficients non constants. Etant donné que l'équation pour u est considérée comme intégrée, on dira qu'il en est de même de l'équation de Riccati initiale.

Ces différents exemples permettent déjà de percevoir la difficulté de définir l'intégrabilité d'une manière générale pour un système dynamique régi par des O.D.E. Nous serons confrontés à ce problème tout au long de cette thèse et serons aussi amenés à admettre comme intégrables les systèmes dont la résolution se ramène à un problème linéaire au moyen d'une transformation locale ou d'une équation transcendante de Painlevé. Ces transcendantes peuvent, en effet, se ramener à des équations intégrales linéaires par des changements de variables non locaux.

Ces différentes manières de décrire l'intégrabilité peuvent sembler disparates à première vue et l'étude de l'intégrabilité apparaît alors une tâche gigantesque. Il faut cependant noter que c'est la connaissance d'une ou plusieurs intégrales du mouvement qui permet de progresser dans cette étude. En effet, même si on ne connaît pas un nombre suffisant de constantes pour prouver l'intégrabilité, on peut toujours les utiliser pour réduire le système. Ceci explique pourquoi la recherche des constantes du mouvement sera une de nos préoccupations essentielles tout au long de cette thèse. Mais cette recherche ne peut généralement pas être implémentée seule. Elle met en oeuvre des calculs trop fastidieux pour pouvoir être entreprise sans guide.

Les trois premiers chapitres de cette thèse sont consacrés à la présentation de trois méthodes qui permettent de détecter si un système est un bon candidat pour l'intégrabilité. Nous présentons aussi les principaux résultats que nous avons obtenus, en lien avec ces méthodes.

Dans le premier chapitre, nous montrons comment les techniques d'*intégration numériques* permettent de repérer les différents types de comportement d'un système dynamique et en particulier de distinguer un système intégrable d'un système chaotique.

Le deuxième chapitre est lui consacré à la présentation de techniques reliées à la théorie des algèbres de Lie. Ces techniques peuvent se révéler très utiles dans l'étude des systèmes différentiels, mais leur domaine d'application semble, malheureusement, limité pour le moment aux systèmes hamiltoniens écrits sous une forme bien spécifique.

La troisième méthode a son origine dans les travaux de Painlevé et de Kowalevskaya à la fin du siècle dernier. L'idée sous-jacente est la suivante :

Considérons une équation différentielle ordinaire. Dès que les coefficients sont suffisamment réguliers, on peut intégrer localement cette équation à partir d'une condition initiale donnée. Le problème de l'intégration est en fait celui d'étendre cette solution locale en une solution globale. On peut espérer raisonnablement que si les singularités critiques de l'équation sont fixes, cette extension sera possible. C'est cette idée qui est à la base des travaux de Kowalevskaya puis Painlevé pour relier l'intégrabilité à l'analyse des singularités des O.D.E. Dans le chapitre 3, nous présentons les principales étapes qui ont conduit à l'élaboration du critère de Painlevé sous sa forme actuelle.

A ce point, il faut noter que tous ces moyens de détection ne constituent pas en eux-mêmes une preuve d'intégrabilité. Ils ne permettent que de sélectionner les systèmes potentiellement intégrables. Le calcul explicite des constantes du mouvement fournit une preuve d'intégrabilité.

Les deux chapitres suivants traitent des systèmes hamiltoniens et dissipatifs. Nous y présentons l'étude d'un certain nombre de systèmes dynamiques au point de vue de l'intégrabilité et mettons en évidence le va-et-vient constant qui apparaît entre l'utilisation du critère de Painlevé et la recherche directe de constantes du mouvement.

Cette méthode, qui est systématique dans le cas hamiltonien indépendant du temps, s'étend de manière naturelle au cas où les hamiltoniens dépendent explicitement du temps. Ces résultats sont présentés dans le chapitre 6.

Les mêmes questions d'intégrabilité, de chaos ... se posent aussi pour les systèmes quantiques, mais les notions sont moins bien définies et les problèmes encore assez mal cernés. Dans le chapitre 7, on examine comment on peut, à partir de systèmes classiquement intégrables, obtenir des systèmes quantiques qui possèdent la même propriété.

Dans le chapitre 8, c'est la notion de chaos quantique que l'on cherche à étudier. Pour cela, on examine numériquement un modèle dont l'homologue classique est bien connu pour son comportement chaotique.

CHAPITRE 2

UTILISATION DES TECHNIQUES NUMÉRIQUES POUR L'ÉTUDE DES SYSTÈMES DYNAMIQUES

A - INTRODUCTION

Lorsque l'on est en présence d'un système dynamique régi par des O.D.E., on cherche généralement à déterminer le comportement asymptotique du système. En fait, on aimerait pouvoir prédire, à partir de conditions initiales spécifiées, l'état du système au bout d'un temps plus ou moins long (long par rapport à un temps caractéristique du système). Comme on l'a dit dans l'introduction générale, cela est possible lorsque les équations qui modélisent le système sont intégrables. Mais, même dans ce cas particulier, on ne sait en général pas exprimer les solutions sous forme "fermée". C'est seulement pour des classes très limitées d'équations que l'on connaît des méthodes analytiques d'intégration. D'ailleurs, même si on a une forme explicite pour les solutions, elle peut être si compliquée que les propriétés qualitatives ne sont pas visibles.

Ceci explique pourquoi, dans la plupart des cas, on est obligé de faire appel à des techniques de calcul approché pour avoir une idée du type de comportement des solutions.

Parmi d'autres, le calcul numérique (digital ou analogique) fournit un outil puissant d'investigation.

On utilise souvent des schémas de discrétisation numériques pour intégrer de manière approchée les équations du mouvement. Le temps est alors considéré comme un paramètre discret et il suffit de laisser le schéma évoluer pendant suffisamment longtemps pour obtenir une information sur l'évolution du système. Malheureusement, même cette technique ne permet pas toujours de donner une réponse au problème de prédictabilité à partir des conditions initiales.

Certains types de comportement, appelés "chaotiques", sont justement caractérisés par cette propriété de manque de prédictibilité à long terme. Bien que le système soit déterministe, après un temps assez long, de minuscules erreurs dans l'évaluation ou la spécification de l'état initial peuvent grandir de manière incontrôlée. Par conséquent, toute solution approchée du système, obtenue en discrétisant le temps, s'éloigne très vite, voire exponentiellement du comportement réel du système.

Un bon exemple en est donné par l'application suivante (où le temps est discrétisé) :

$$x_{t+1} = 4x_t(1-x_t) \qquad (0 < x_t < 1)$$

Sa solution explicite s'écrit :

$$x_t = \sin^2 \pi u_t \text{ où } u_t = 2u_{t-1} \pmod 1$$

c'est-à-dire que si $u_0 = b_0\, b_1 \ldots\, b_t$ en notation binaire, alors $u_t = b_t$. On vérifie de suite que si l'état initial u_0 est connu avec t chiffres en binaire, u_t est complètement inconnu.

Il apparaît donc, à ce point, que les méthodes numériques ne pourront pas donner des informations exactes sur le comportement des systèmes chaotiques. Elles permettent tout de même d'étudier la nature du chaos et les différentes routes qui y conduisent. Par contre, lorsque le système est intégrable, son comportement est régulier et, en intégrant numériquement les équations du mouvement, on peut distinguer un tel système d'un système chaotique.

Encore faut-il pour cela que le mouvement soit borné. Sinon cette méthode permet effectivement d'affirmer que le système part à l'infini mais donne peu d'indications sur la nature du mouvement et sur la régularité des trajectoires. Pour les systèmes hamiltoniens, l'étude du potentiel V permet quelquefois de déterminer les régions de l'espace des phases où le mouvement restera borné. Dans les autres cas, on a quelquefois recours à une analyse du système au voisinage de ses points critiques, obtenue en linéarisant localement le système. Cette méthode ne donne pas toujours une idée du comportement global du système mais elle permet, dans certains cas, de prédire les zones où le mouvement restera borné.

En pratique, nous avons surtout utilisé les techniques d'intégration numérique pour nous aider à identifier les systèmes intégrables en dimension 2 ou 3.

Pour cela, il est utile de connaître, au moins grossièrement, les différents types de comportement qu'un système dynamique non linéaire peut exhiber. Nous en donnons une présentation rapide dans la suite, illustrée par des exemples concrets.

B - ETUDE NUMERIQUE DES SYSTEMES DYNAMIQUES

Dans cette partie, nous nous proposons de montrer sur des exemples comment les techniques d'intégration numérique permettent de repérer les différents types de comportements possibles d'un système. Comme il a été dit dans l'introduction, il faut surtout s'attendre ici à des résultats qualitatifs concernant les trajectoires du système.

Différentes méthodes existent pour intégrer numériquement les équations du mouvement mais nous avons toujours employé un schéma de Runge et Kutta à l'ordre 4 qui présente le grand avantage d'être à la fois simple et suffisamment précis.

Une fois les équations intégrées numériquement, se pose le problème de la visualisation des trajectoires. Dès que le système différentiel est de dimension supérieure à 2, on ne peut pas représenter exactement les trajectoires sur le plan. On a alors recours soit à des projections sur différents plans de coordonnées, soit à l'utilisation de surfaces de section. Cette dernière technique permet de ramener l'étude d'un système différentiel de dimension N à celle d'une application de dimension N-1 construite de la façon suivante :

Considérons un système différentiel de la forme :

$$\dot{x}_i = f_i(x_1,\ldots,x_N) \qquad (i = 1,2,\ldots,N) \qquad (1)$$

Une solution particulière de *(1)* engendre une orbite dans l'espace des phases. On choisit alors une surface (la surface de section) de dimension N-1 et on étudie les intersections de cette orbite avec la surface. On définit ainsi une application sur la surface de section ; la section de Poincaré est l'image de cette application sur cette surface.

Comme nous allons le voir par la suite, on utilise surtout les sections de Poincaré pour étudier les systèmes hamiltoniens de dimension 2.

C - SYSTEMES HAMILTONIENS A DEUX DIMENSIONS

Etant donné un hamiltonien :

$$H = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + V(x,y)$$

les équations de mouvement ont la forme suivante :

$$\begin{cases} \ddot{x} = -\dfrac{\partial V}{\partial x} \\ \ddot{y} = -\dfrac{\partial V}{\partial y} \end{cases}$$

L'espace des phases est alors de dimension 4 correspondant aux variables $(x,\dot{x},y,\dot{y})$; cependant, comme l'énergie totale est une constante du mouvement, ces quatre variables ne sont pas indépendantes et on peut, par exemple, exprimer $\dot{y}$ en fonction de x et $\dot{x}$.

Considérons le plan $y = 0$ comme surface de section.

La section de Poincaré correspondante est alors décrite dans le plan $(x,\dot{x})$. Si le système est intégrable, c'est-à-dire si il existe une seconde constante du mouvement indépendante de l'hamiltonien H, les trajectoires sont régulières, tracées sur un tore. La section de Poincaré est donc formée d'une courbe plane fermée, régulière.

Cette remarque permet de mettre en évidence les systèmes non intégrables.

En effet, si le système présente une surface de section formée de points dispersés, apparemment chaotique, il y a de fortes chances que le

système ne soit pas intégrable. Nous allons voir à présent sur un certain nombre d'exemples comment fonctionne ce détecteur.

Considérons le potentiel quartique suivant :

$$V = y^4 + \frac{3}{4} y^2x^2 + \frac{x^4}{d}$$

Une étude précédente, basée sur l'étude des singularités du système, nous a montré que les valeurs d = 8, d = 16 et d = $\frac{1\ 088}{3}$ étaient des candidats possibles d'intégrabilité (voir chapitre 4).

Pour d = 16, ce potentiel est bien intégrable avec une seconde constante quadratique dans les vitesses. La figure 1 montre une surface de section pour différentes valeurs de l'énergie.

Pour d = 8, nous avons aussi été capables de calculer une seconde constante du mouvement qui se trouve être quartique dans les vitesses.

Le cas d = $\frac{1\ 088}{3}$ est différent. Non seulement, il n'a pas été possible de calculer une deuxième constante du mouvement jusqu'à l'ordre 6, mais les surfaces de section présentent une nette apparence de chaos *(figure 2)*.

Bien entendu, on ne peut pas exclure la possibilité qu'il existe une constante d'un ordre très élevé qui donne lieu à une courbe fermée compliquée du type : "pelote de laine ayant subi les tortures d'un chat". Cependant, le nombre des points sur la surface de section montre que l'ordre de la constante devrait être autour du millier ; l'existence d'une telle constante paraît donc nettement improbable.

Nous avons aussi étudié l'effet d'une perturbation sur le potentiel intégrable $V = y^4 + \frac{3}{4} y^2x^2 + \frac{x^4}{8}$. Si on ajoute à ce potentiel un terme de la forme λx ou λx^2, la propriété de Painlevé (voir chapitre 4) n'est plus satisfaite. Dans le premier cas, des puissances demi-entières vont entrer dans le développement en série d'une solution, détruisant le caractère Painlevé. Les surfaces de section *(figure 3)* de ce potentiel présentent une très nette apparence de chaos, signe d'une probable non-intégrabilité.

Si, par contre, on perturbe le potentiel par un terme en λx^2, des logarithmes entrent dans le développement, de nouveau le caractère Painlevé est détruit et on s'attend à un comportement non-intégrable. Les sections de Poincaré, pour différentes conditions initiales, paraissent régulières *(figure 4)* tant qu'elles ne se rapprochent pas de l'origine. Par contre, la figure 5 montre trois sections qui passent près de l'origine. Le comportement au voisinage de ce centre est rendu plus clair par un grossissement des sections précédentes *(figure 6)*. La présence effective de chaos ne laisse plus de doute.

Nous aimerions à présent attirer l'attention sur le type de difficultés que l'on peut rencontrer au cours de telles études numériques et sur les précautions qu'il convient alors de prendre.

Figure 1
Surfaces de section du potentiel intégrable :
$V = y^4 + \frac{3}{4} y^2 x^2 + \frac{x^4}{16}$
pour différentes valeurs de l'énergie
(E = 0,1 ; 0,05 ; 0,01)

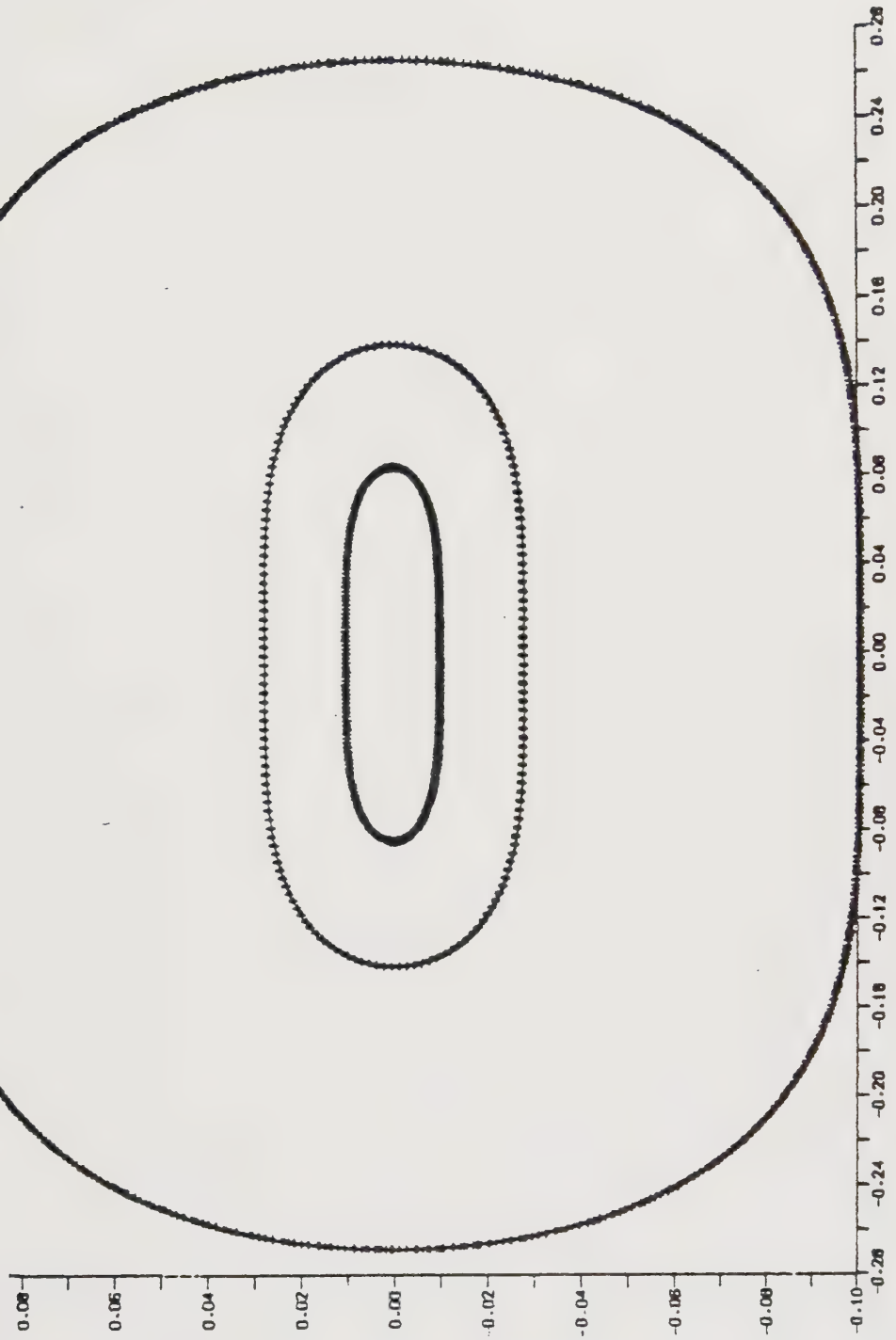


Figure 2
Surface de section du potentiel :
$V = y^4 + \frac{3}{4} y^2 x^2 + \frac{x^4}{1088} - \frac{3}{1088}$
(pas intégrable)

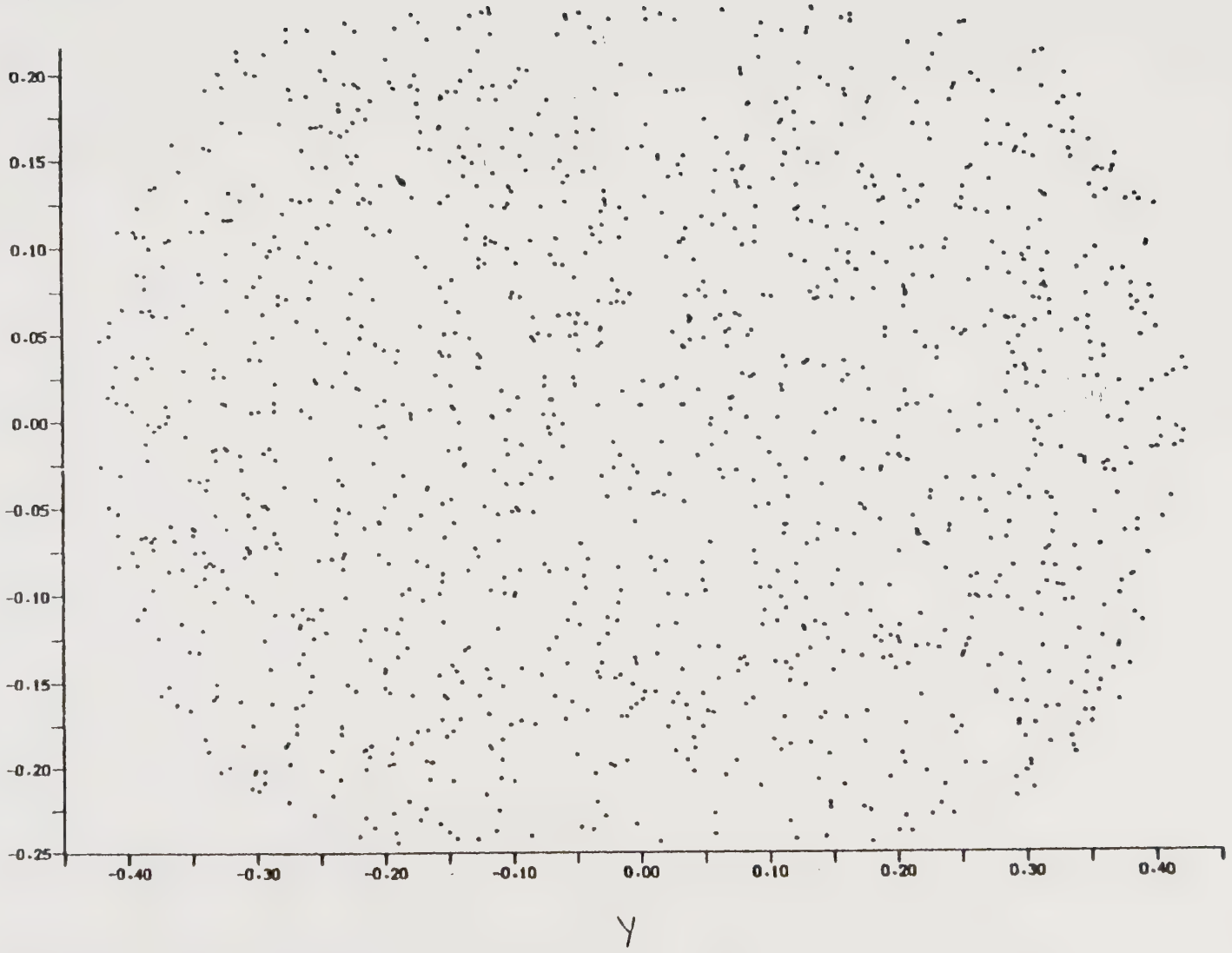


Figure 3

$V = y^4 + \frac{3}{4} y^2 x^2 + \frac{x^4}{8} + x$

Surfaces de section pour différentes conditions initiales

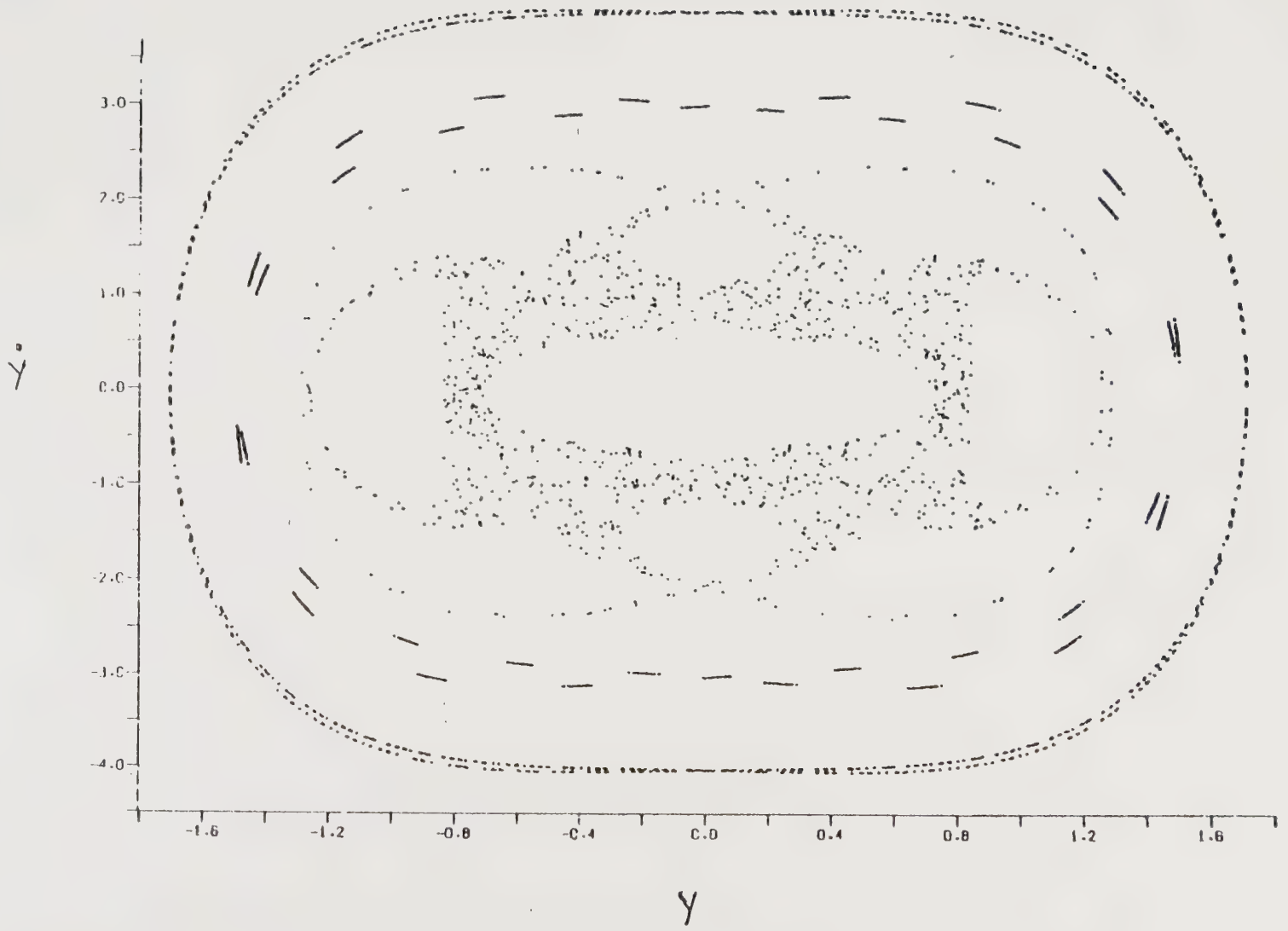
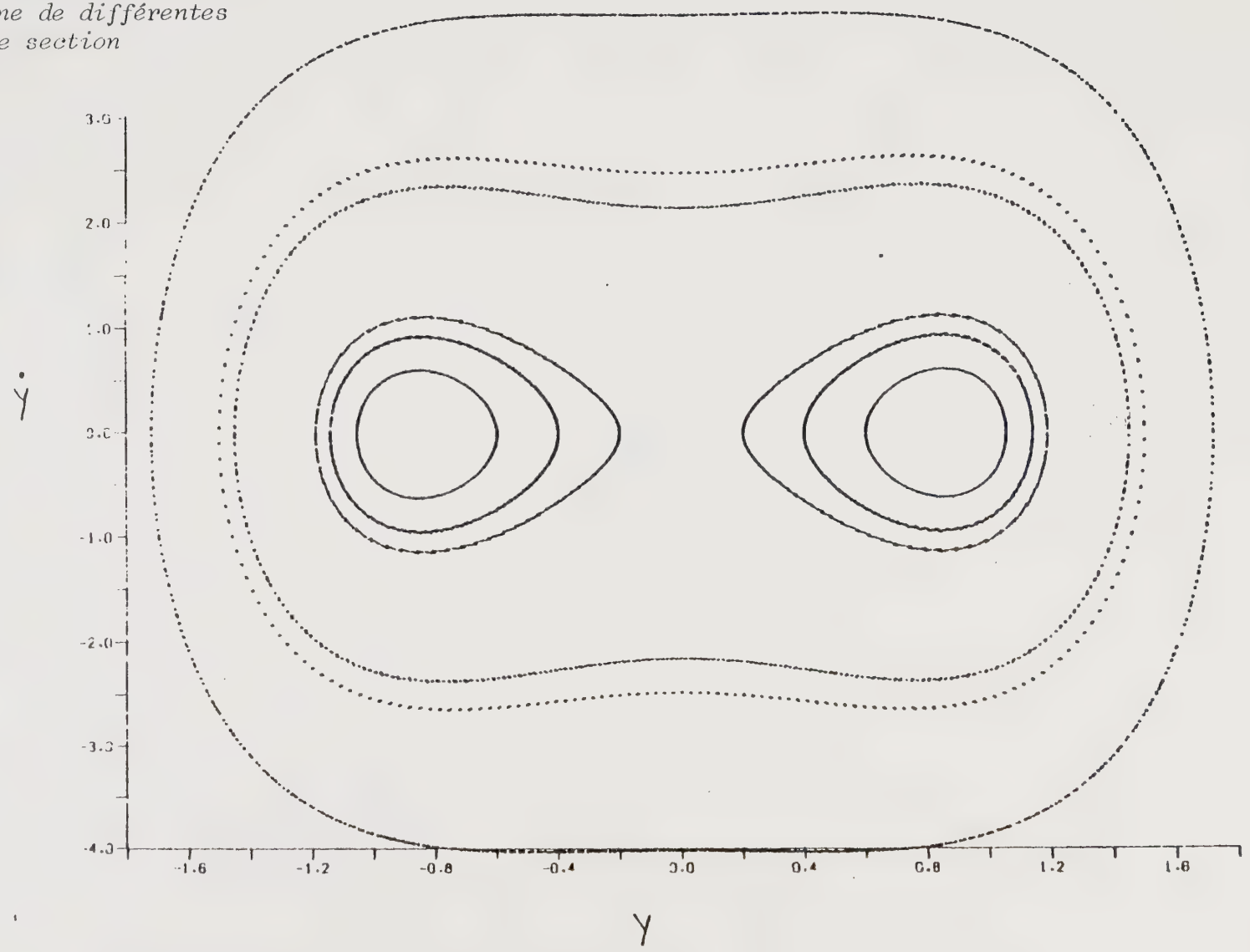


Figure 4

$V = y^4 + \frac{3}{4} y^2 x^2 + \frac{x^4}{8} + x^2$

Apparence régulière, loin de l'origine de différentes surfaces de section



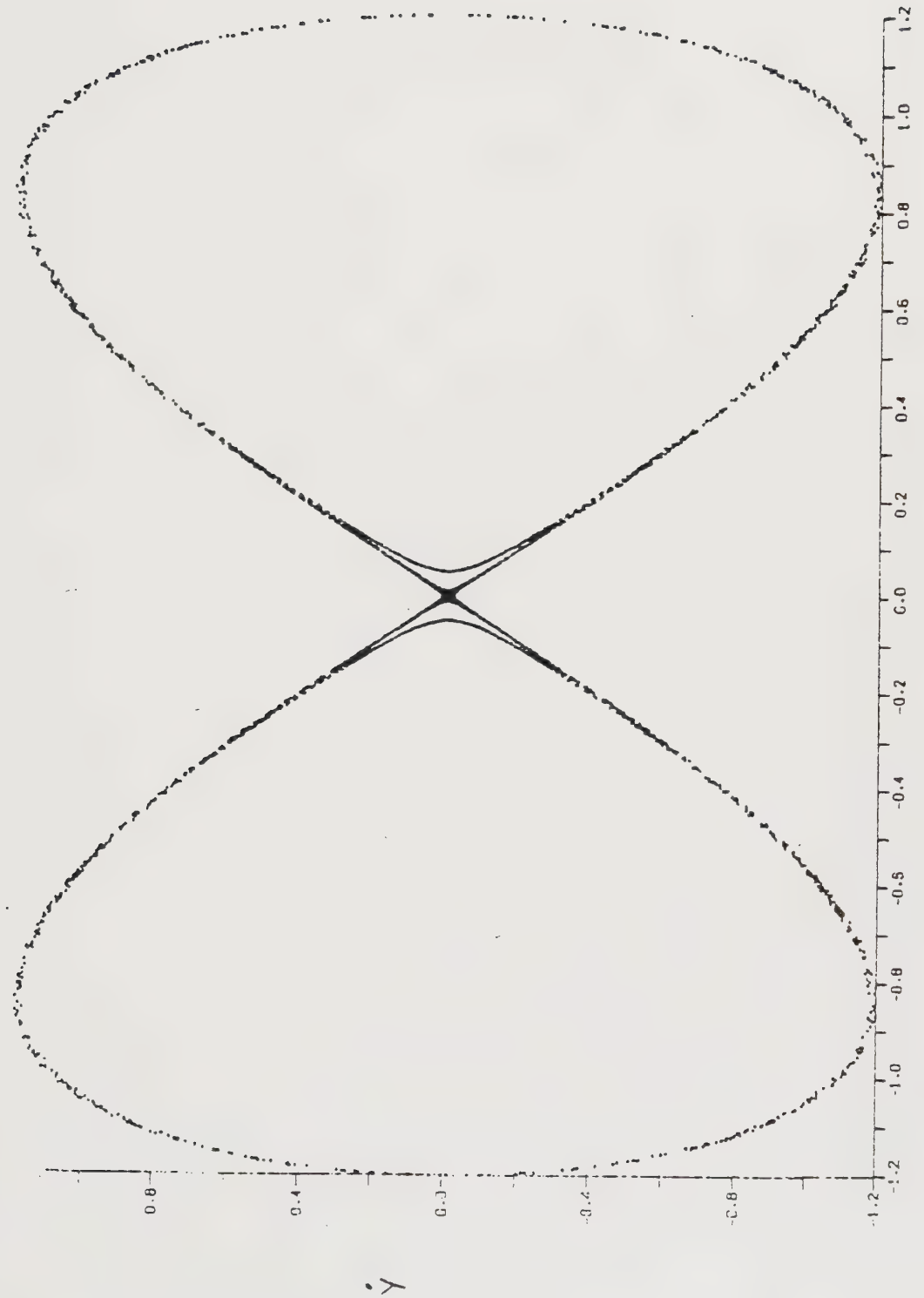
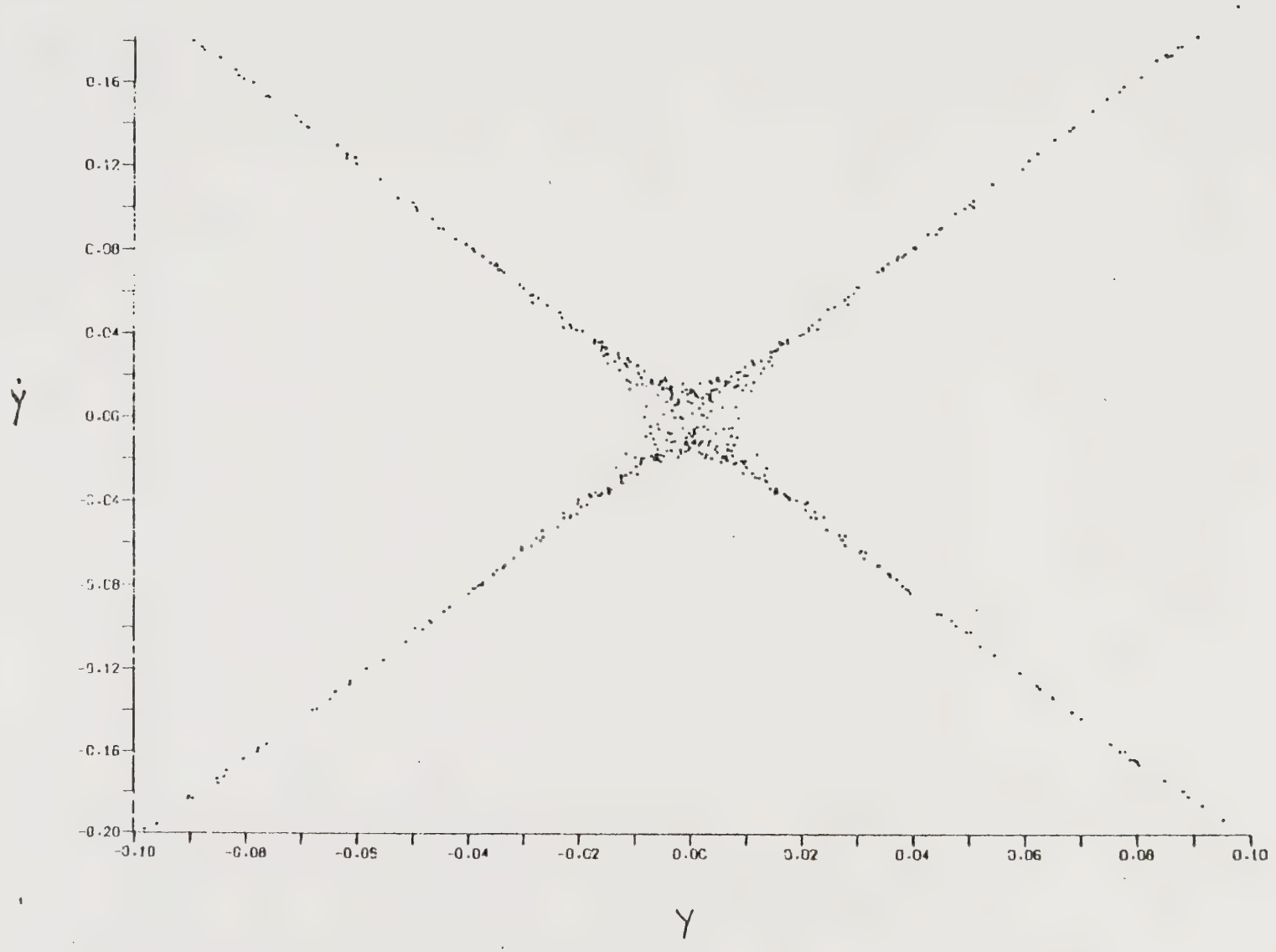


Figure 5
Sections passant par l'origine

Figure 6
Grossissement autour de l'origine de la section précédente



Tout d'abord, on ne peut pas se contenter d'étudier le système pour une valeur fixée de l'énergie. En effet, il arrive souvent que le système ait une allure intégrable pour les petites ou les grandes valeurs de l'énergie et ne présente du chaos manifeste que pour les valeurs intermédiaires. Il faut donc faire varier l'énergie dans les plages autorisées par le système et de même faire varier les conditions initiales à énergie fixée.

D'autre part, une apparence de chaos peut être due non pas au comportement réel du système mais à un manque de précision numérique. Pour contrôler ces instabilités, on s'attache à vérifier que l'énergie totale du système est conservée avec une bonne précision au cours des itérations.

Le potentiel suivant fournit un bon exemple de cette situation :

$$V = y^2 x^{-2/3} + \frac{3\lambda}{4} x^{4/3} + \mu x^{-2/3}$$

Ce système a été prouvé intégrable pour $\lambda = 1$ par Holt [1], et pour $\lambda = 6$ et $\lambda = 16$ par nous et par Hietarinta [2] indépendamment. Le comportement du système est très régulier à petites énergies et ceci que λ corresponde à un cas intégrable ou pas. Pour mettre en évidence l'apparition de chaos, il faut alors intégrer avec des énergies assez élevées. Mais, dans ces conditions, le terme $y^2 x^{-2/3}$ crée, lorsque x prend des valeurs petites, de grands différentiels dans la vitesse qui du fait de la discrétisation donnent lieu à des sauts dans les valeurs de l'énergie.

Une nette apparence de chaos en résulte même dans les cas intégrables.

La figure 7 montre une section pour $\lambda = 1$, une énergie $E = 5$, et un pas d'itération $H = .02$. Dans ce cas, l'énergie est assez mal conservée ; tandis que pour les mêmes valeurs de l'énergie et du paramètre, mais un pas plus petit ($H = .01$), on retrouve l'image régulière de l'intégrabilité *(figure 8)*.

Une autre difficulté provient de ce qu'un système voisin d'un système intégrable a un comportement peu différent et semble donc régulier sans pour autant être lui aussi intégrable. Des sections de Poincaré régulières ne peuvent en aucun cas être une preuve d'intégrabilité du système. Par contre, une apparence de chaos garantit la non intégrabilité, à partir du moment où l'énergie est bien conservée.

Ali et Somorjai [3] ont étudié le système formé de trois particules de masses égales interagissant au moyen de potentiels de Morse :

$$V_\sigma(r) = \exp(-2br) - 2\sigma \exp\left(-\frac{b}{\sigma}\right) \qquad (\sigma > 0)$$

avec des conditions aux limites périodiques. Les surfaces de section de ce système sont régulières pour des valeurs différentes de l'énergie et aucune trace de chaos n'a pu être mise en évidence.

Néanmoins, l'analyse des singularités du système montre que la propriété de Painlevé n'est jamais satisfaite pour $\sigma \neq 0$.

En conséquence, il y a de fortes présomptions pour que ce système ne soit pas intégrable. La régularité des sections de Poincaré doit être simplement due au fait que, à la limite $\sigma = 0$, ce système se confond

Figure 7
Potentiel :
$V = y^2 x^{-2/3} + \frac{3}{4} x^{4/3} + \mu x^{-2/3}$
à l'énergie E = S. et un pas d'itération H = .02, nette apparence de chaos.

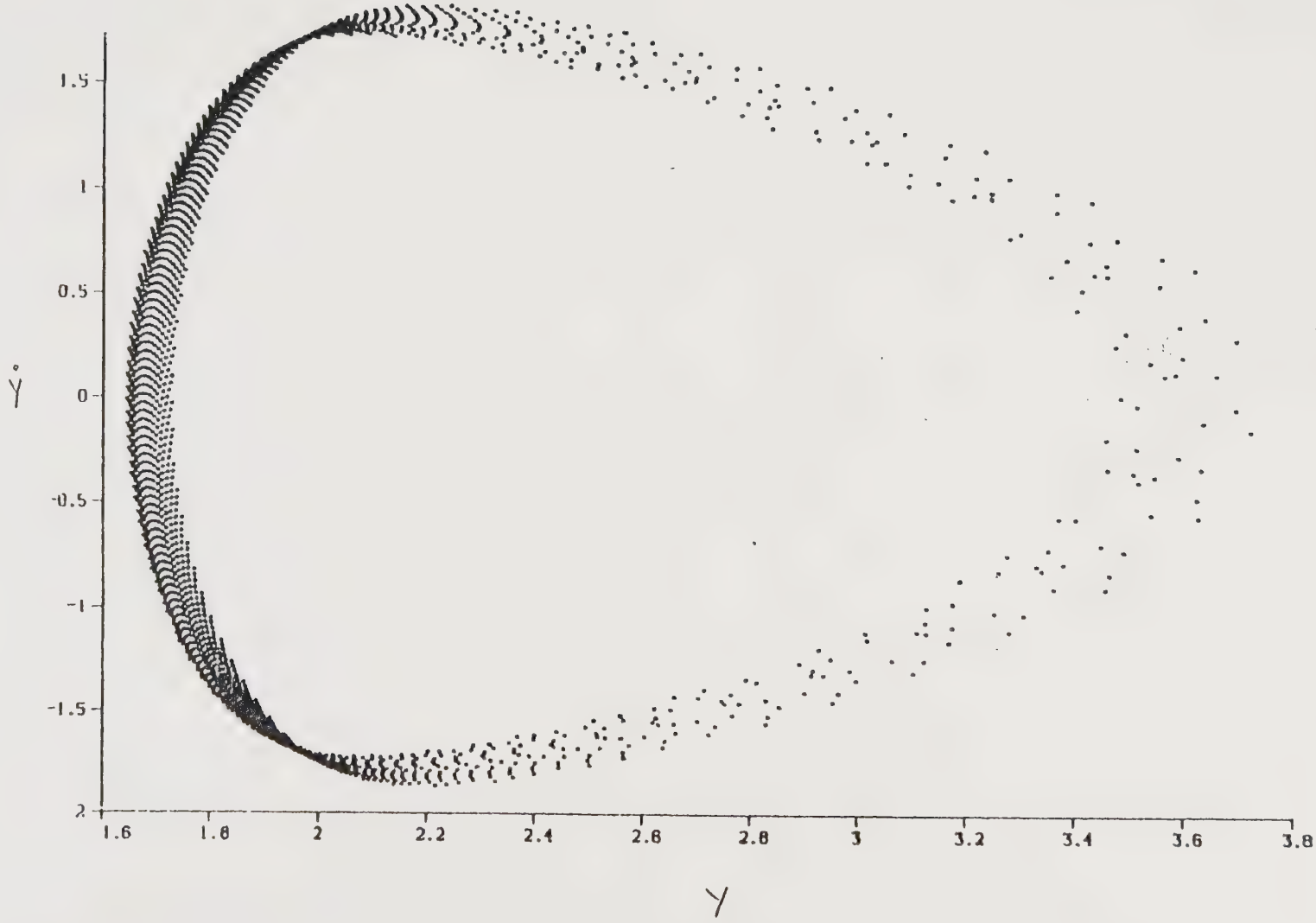
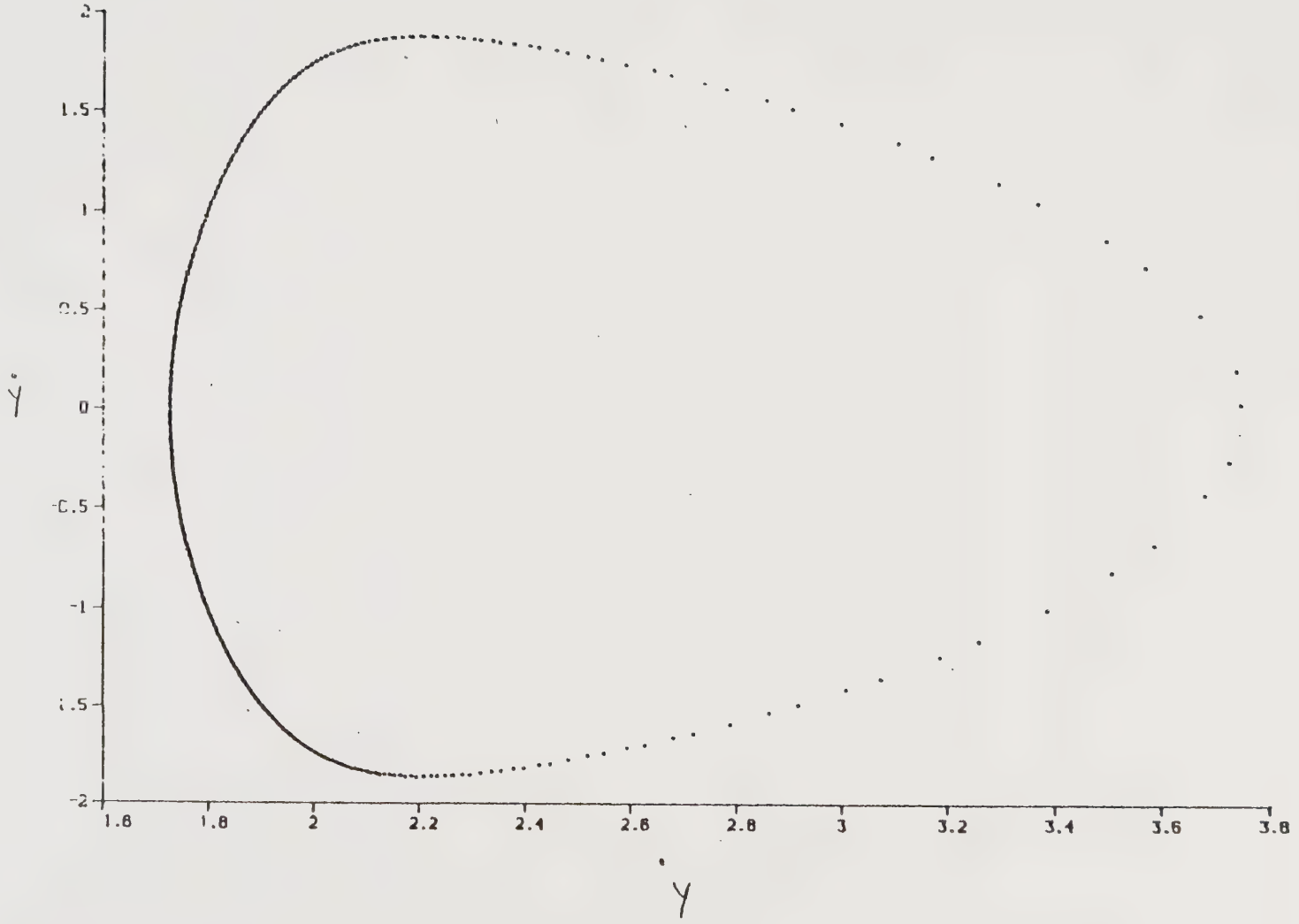


Figure 8
Même potentiel que dans le cas de la figure 7, à la même énergie avec un pas H = .01.



avec le système de Toda périodique à trois particules qui lui est bien connu comme intégrable.

D - SYSTEMES DISSIPATIFS

Nous allons tout d'abord considérer des systèmes dissipatifs de dimension 2.

Dans ce cas, l'analyse linéaire autour des points critiques permet déjà d'avoir une idée du comportement du système et surtout de délimiter des régions de l'espace des phases où le mouvement restera borné. Mais cette analyse peut s'avérer insuffisante lorsque les équations sont non-linéaires.

En effet, lorsque le point critique est associé à des valeurs propres imaginaires pures, les trajectoires *d'un système linéaire* autour de ce point sont fermées (c'est un centre). Par contre, ce n'est plus vrai pour un système non linéaire car une perturbation, même petite, d'une orbite fermée, peut donner naissance à une trajectoire ouverte. Il s'en suit que l'on rencontre, dans le cas non-linéaire, des trajectoires soit ouvertes, soit fermées autour d'un tel point. Dans le dernier cas, le système est probablement intégrable et il existe généralement une constante du mouvement indépendante du temps qui permet au moins théoriquement de ramener les équations du mouvement à des quadratures.

Les deux systèmes suivants illustrent ces deux situations :

$$\begin{cases} \dot{x} = -x^2 + 2xy + x \\ \dot{y} = -y^2 + 5xy - 2y \end{cases} \qquad (1)$$

$$\begin{cases} \dot{x} = -x^2 + 2xy + x \\ \dot{y} = -y^2 + 5xy + \frac{x}{8} - 3y \end{cases} \qquad (2)$$

Le système *(1)* possède un point fixe associé à des valeurs propres imaginaires pures : $x_0 = \frac{1}{3}$, $y_0 = -\frac{1}{3}$, et une constante du mouvement indépendante du temps a pu être calculée.

L'allure des orbites fermées dans le plan (x,y) est donnée dans la figure 9 pour différentes valeurs des conditions initiales. Elles permettent de caractériser ce point critique comme un centre.

Le système *(2)* a lui aussi un point fixe de valeurs propres imaginaires pures pour $x_0 = \frac{2}{3}$, $y_0 = -\frac{1}{6}$. La figure 10 permet de se convaincre que l'on a affaire à un point spirale instable et non à un centre. En plus, ce système ne possède pas la propriété de Painlevé (cf. chapitre 4) et aucune constante du mouvement n'a pu être calculée.

Dans ce dernier cas, le système part en fait à l'infini. On trouve là l'un des deux types d'attracteurs que l'on peut rencontrer en dimension 2 : les points fixes à distance finie ou infinie et les cycles limites. Par contre, en dimension 3, il existe d'autres types d'attracteurs appelés "étranges", caractéristiques d'un mouvement chaotique.

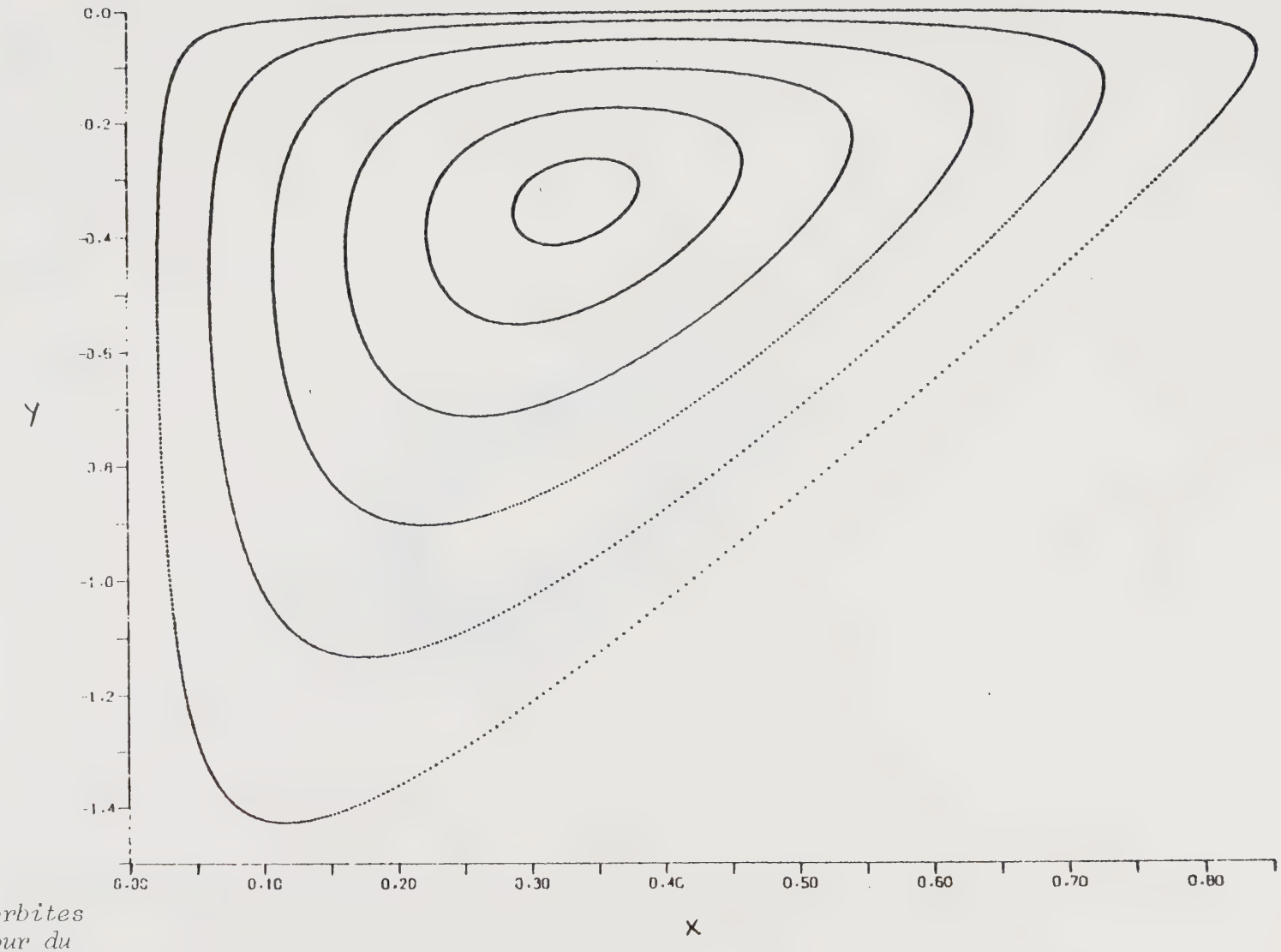


Figure 9
Allure des orbites fermées autour du centre du système :
$\dot{x} = -x^2 + 2xy + x$
$\dot{y} = -y^2 + 5xy - 2y$
$(x_0 = \frac{1}{3}\ ,\ y_0 = -\frac{1}{3})$

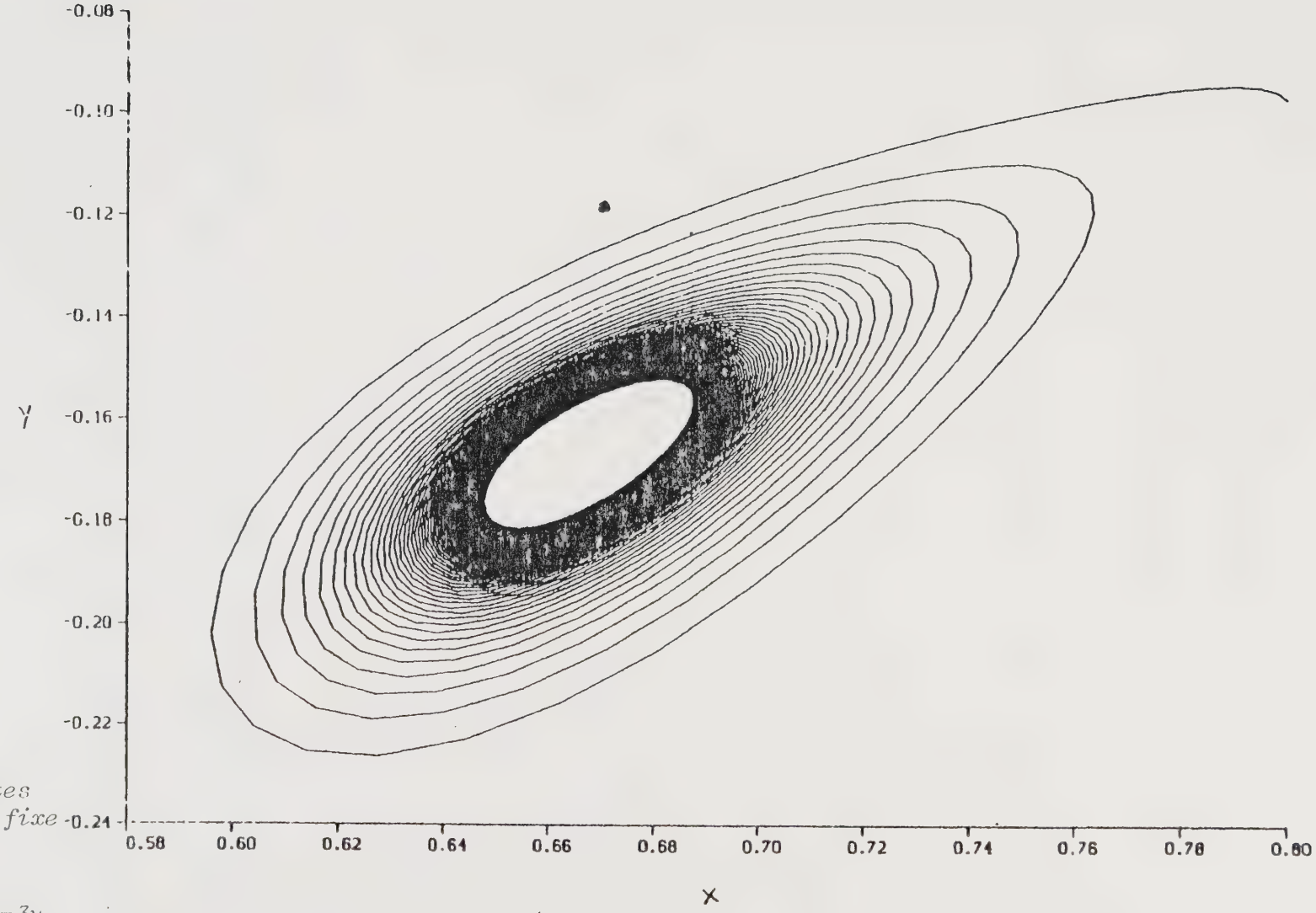


Figure 10
Allure des orbites autour du point fixe du système :

$\dot{x} = -x^2 + 2xy + x$

$\dot{y} = -y^2 + 5xy + \frac{x}{8} - 3y$

$(x_0 = \frac{2}{3}\ ;\ y_0 = -\frac{1}{6})$

Point spirale instable.

Pour de tels systèmes, l'étude des points critiques n'est plus d'un grand recours et on utilise surtout les différentes projections sur les plans de coordonnées.

Nous allons illustrer sur deux modèles classiques certains types de comportement spécifiques à la dimension 3 sans chercher à être exhaustif en aucune manière.

Le premier modèle considéré est celui que Rikitake a proposé pour décrire l'évolution séculaire du dipole terrestre qui présente des renversements aléatoires.

Les équations du mouvement ont la forme suivante :

$$\begin{cases} \dot{x} = yz + \beta x + \mu y \\ \dot{y} = xz + \beta y - \mu x \\ \dot{z} = 1 - xy \end{cases}$$

Lorsque $\mu = \beta = 0$, ce système est intégrable et possède comme constantes :

$$\begin{cases} x^2 - y^2 = D \\ x^2 + y^2 + 2z^2 - 4 \ln(x+y) = E \end{cases}$$

Les figures 11 et 12 présentent les projections du système sur les axes (x,y) et (y,z).

Lorsque $\mu = \beta = \frac{1}{\sqrt{2}}$, le système n'est plus intégrable. Il possède deux points critiques symétriques par rapport à l'axe Oz : $z_0 = -5$, $x_0 = \pm \frac{1}{\sqrt{2}}$, $y_0 = \pm\sqrt{2}$. Le caractère aléatoire des trajectoires du système se présente comme une allée et venue d'un point critique à l'autre (un comportement analogue s'observe aussi pour le système de Lorenz) *(figures 13 et 14)*.

Le dernier exemple que nous allons donner est celui du système de Rössler étudié en détail par Crutchfield *et al.* [4].

Les équations de ce système s'écrivent :

$$\begin{cases} \dot{x} = -(y+z) \\ \dot{y} = x + \alpha y \\ \dot{z} = \alpha + xz - Cz \end{cases}$$

La valeur du paramètre α est fixée à 0.2 et, lorsque le paramètre C varie, le système subit une série de dédoublements de période qui conduisent au chaos [5]. Dans les figures qui suivent, nous présentons certaines étapes de cette route vers le chaos.

Tout d'abord, la projection d'un cycle limite sur le plan (x,y) obtenu pour C = 2.6 *(figure 15)*, puis le dédoublement de ce cycle en quatre périodes *(figure 16)*, et, enfin, pour terminer, une structure en feuilletage de bandes de période 2 *(figure 17)*.

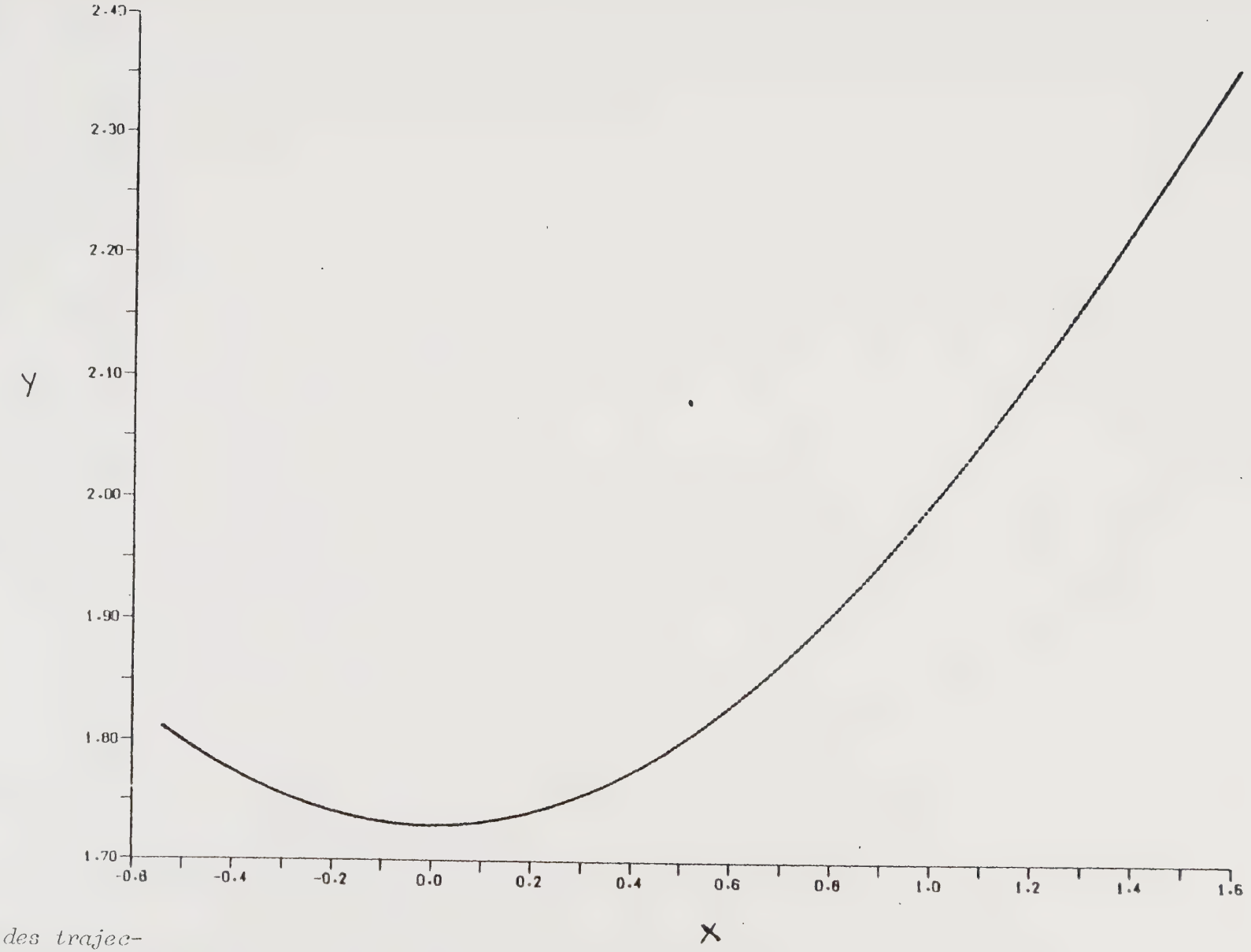


Figure 11
Projection des trajectoires du système de Rikitakee sur le plan (x,y) dans le cas intégrable h = 0, μ = 0.

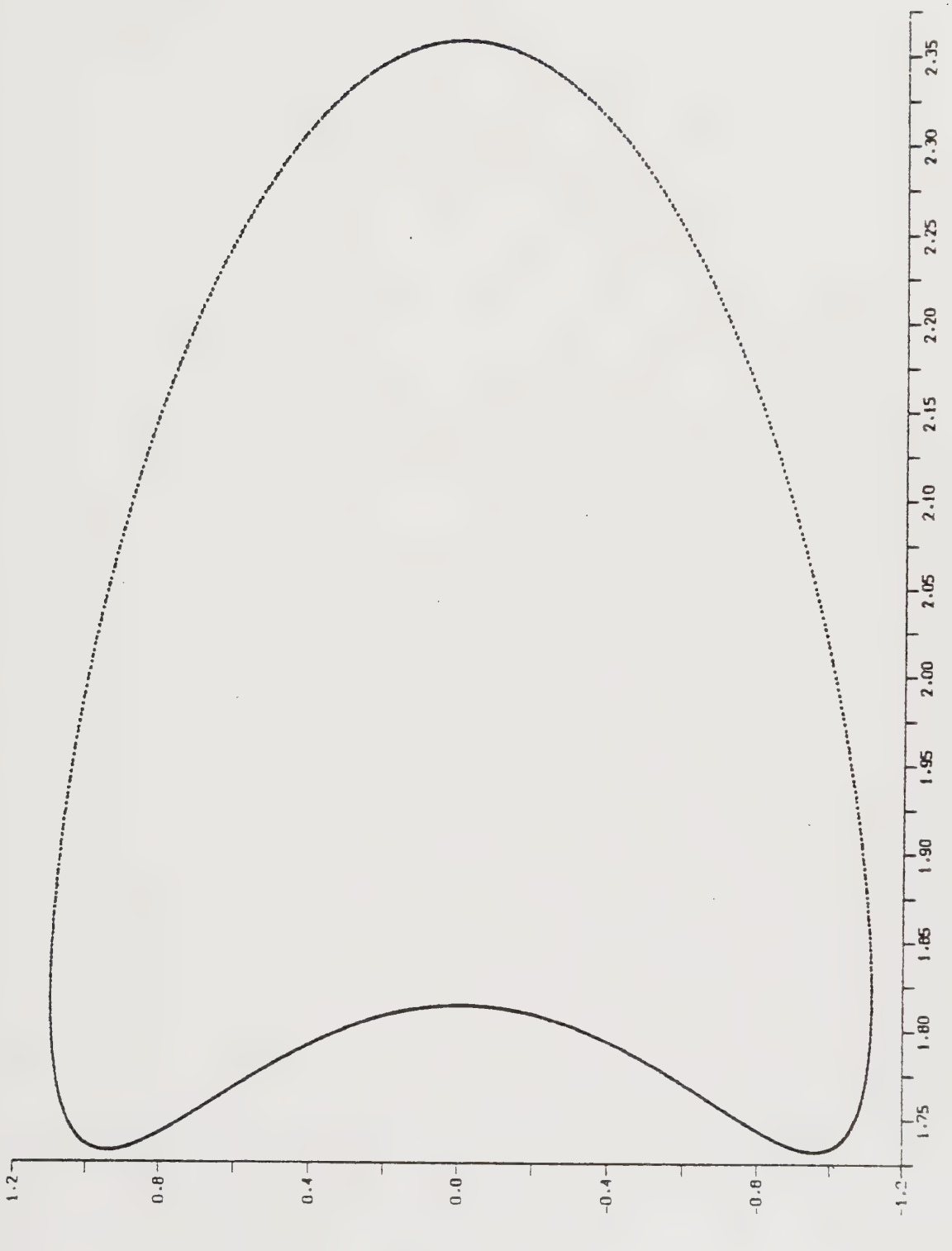


Figure 12
Projection sur le plan de coordonnées (y, z)

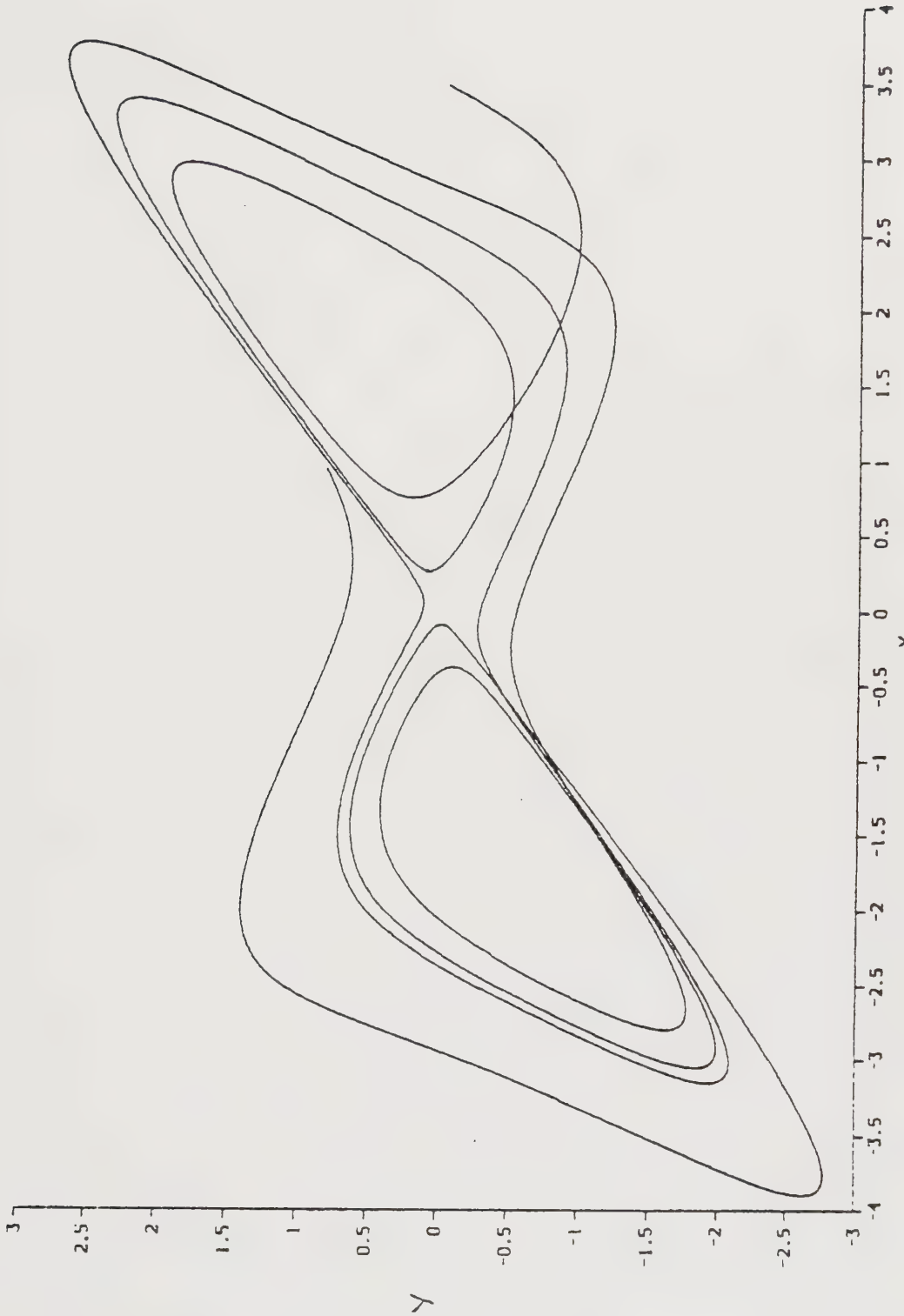


Figure 13
Projection des trajectoires du système de Rikitake dans un cas non-intégrable : sur X, Y.

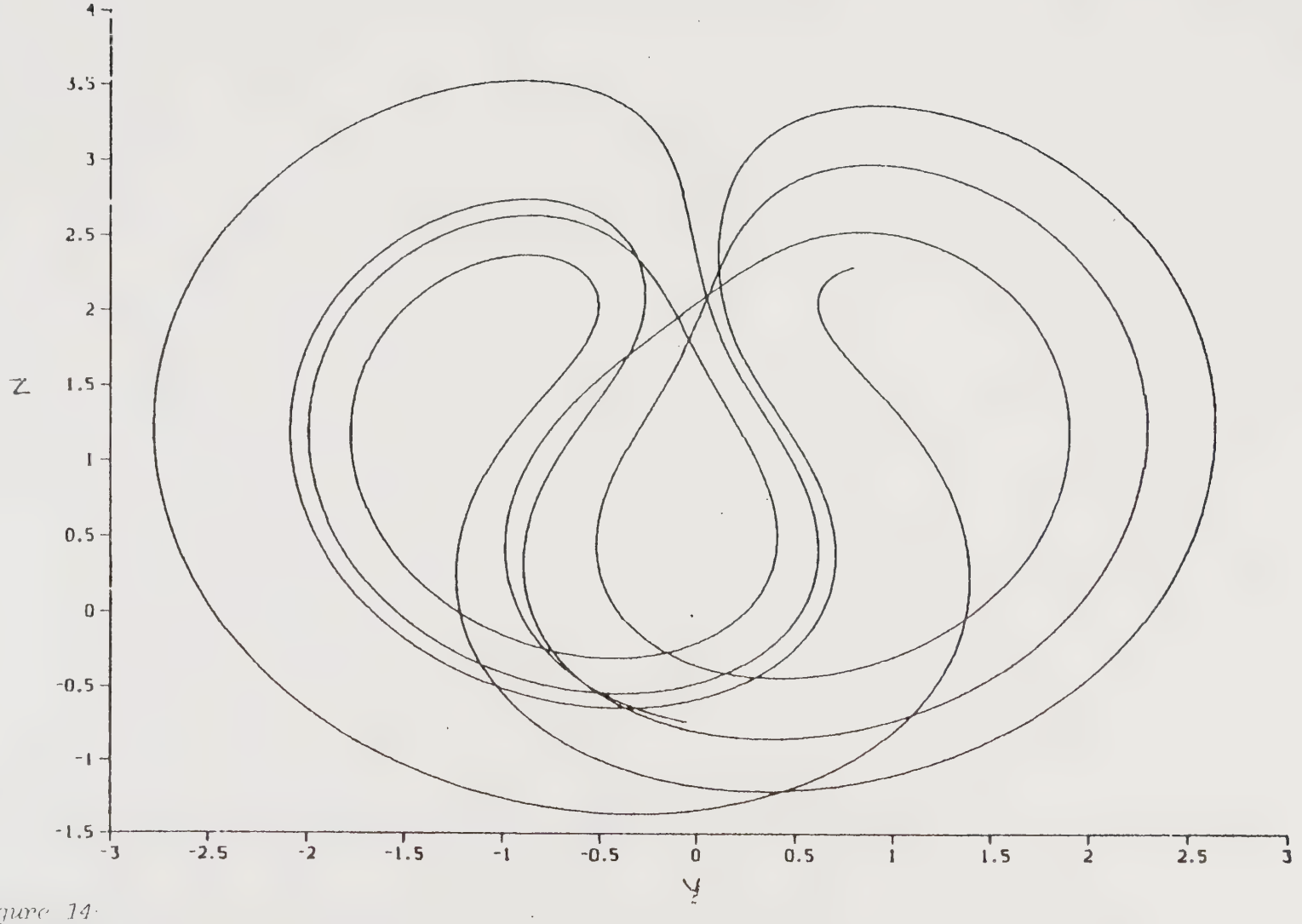


Figure 14
Projection des trajectoires du système de Rikitakee dans un cas non-intégrable : sur Z, Y

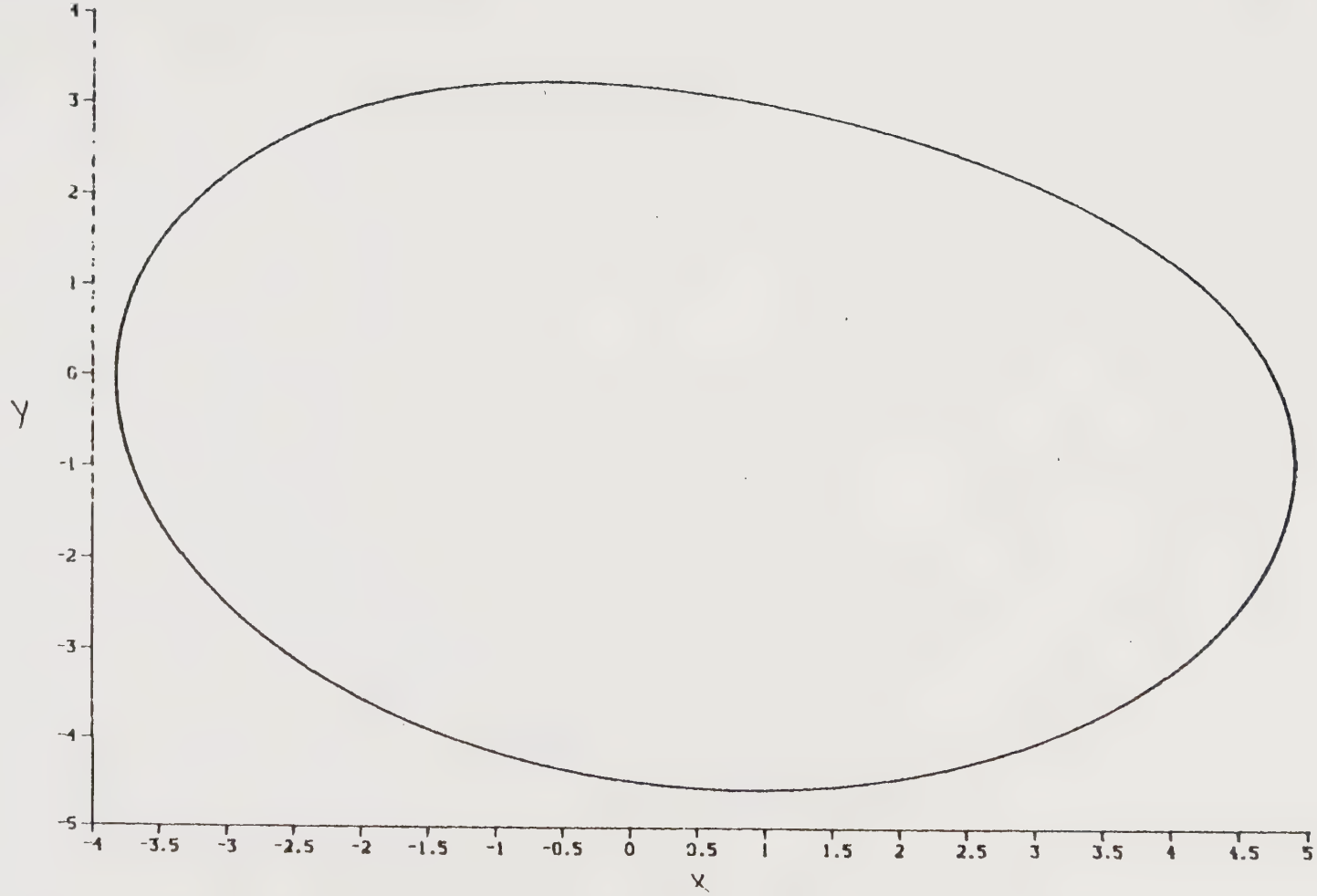


Figure 15
Système de Rössler :
A = .2, B = .2,
C = 2.6
Cycle limite en projection
sur le plan (X,Y).

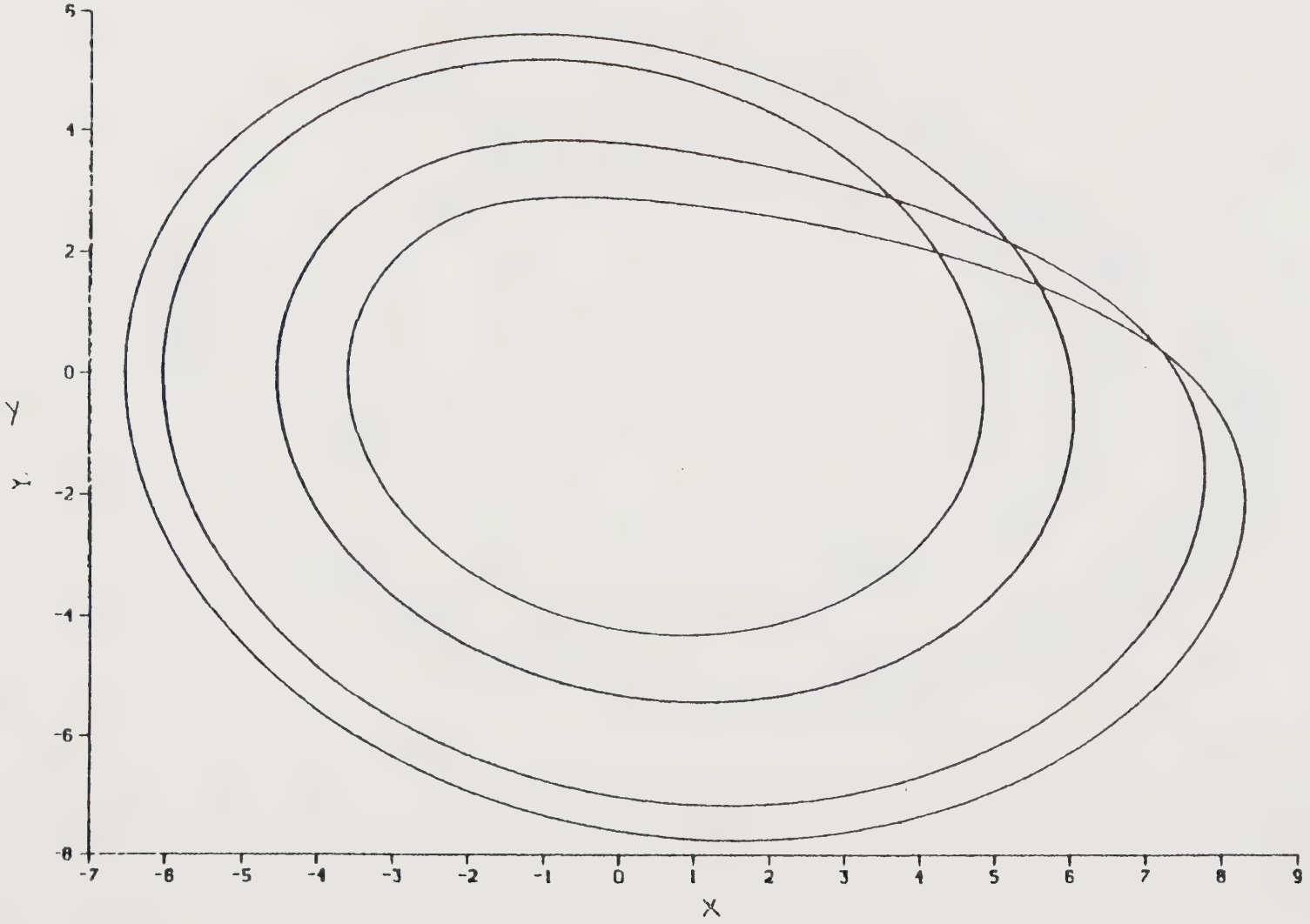


Figure 16
Même système avec C = 4.1.
Dédoublement en quatre périodes.
Projection sur le plan (X,Y).

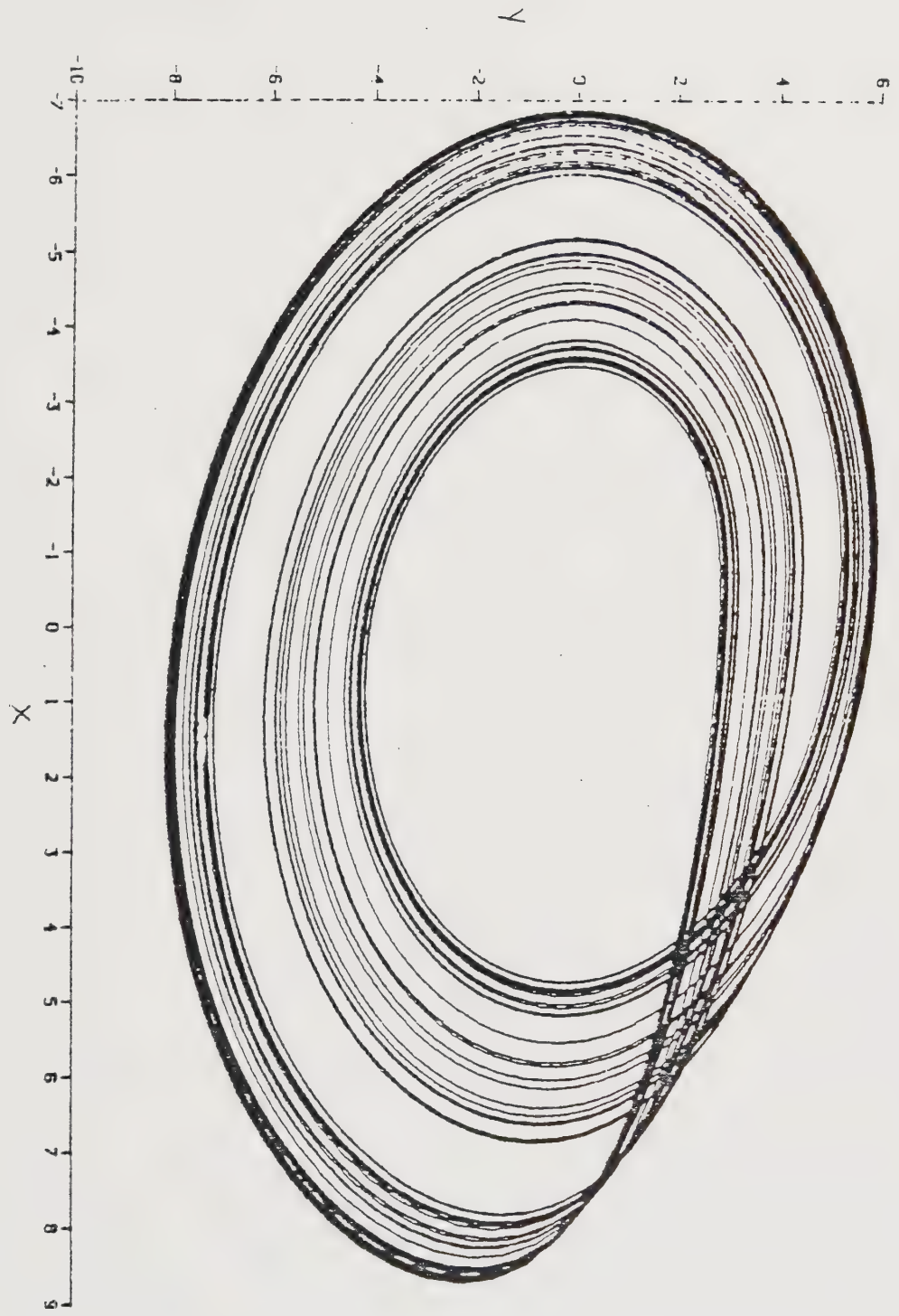


Figure 17
Même système avec C = 4.3

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CHAPITRE 3

UTILISATION DES TECHNIQUES D'ALGÈBRE DE LIE POUR L'ÉTUDE DE L'INTÉGRABILITÉ

C'est d'abord pour l'étude des équations aux dérivées partielles non linéaires que ces techniques ont été développées. Lax [1], un des premiers, eut l'idée de chercher des intégrales de mouvement pour l'équation de Korteweg de Vries comme valeurs propres d'un opérateur linéaire associé L. Cette idée fut reprise par la suite pour les équations de Schrödinger non linéaire, Sine Gordon, etc.

L'idée de base est la suivante :

Considérons une P.D.E. écrite en termes d'une fonction inconnue u. On lui associe deux opérateurs L et H (paire de Lax) dépendants de u, tels que l'évolution :

$$\Psi_t = H\Psi$$

soit compatible avec le problème aux valeurs propres :

$$L\Psi = \lambda\Psi$$

λ est indépendant de t si et seulement si u satisfait à la P.D.E. considérée. De plus, on sait généralement intégrer l'équation non linéaire, c'est-à-dire reconstituer la fonction u à partir des éléments propres de l'opérateur L.

Malheureusement, la détermination d'une paire de Lax pour une équation donnée n'est pas toujours très facile.

Une des premières méthodes que l'on a utilisée pour résoudre les P.D.E. non-linéaires a été de se donner à priori une forme de paire de Lax et de voir ensuite quelles équations elle permettait de résoudre. C'est de cette manière qu'un bon nombre d'équations d'intérêt physique ont été intégrées et classifiées. Malheureusement, par ce moyen, on n'arrive pas à atteindre toutes les P.D.E. non-linéaires. Des équations connues,plus complexes échappent à cette étude et on est obligé actuellement de se poser le problème directement dans les termes suivants : Etant donnée une P.D.E. non linéaire, comment sait-on si elle est intégrable et de quelle méthode dispose-t-on alors pour lui associer une paire de Lax ?

Des recherches récentes ont permis de progresser dans cette direction et commencent à donner des résultats intéressants [2,3].

Dans le cas des systèmes d'équations différentielles ordinaires, nous allons voir que la même difficulté va se présenter [4].

Considérons un système hamiltonien de dimension N. On dit que ce système possède une paire de Lax si il existe deux matrices L et M dont les éléments dépendent des coordonnées q_i et des impulsions p_i ($i = 1,2,\ldots,N$) telles que les équations de mouvement :

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= \frac{\partial H}{\partial q_j} \end{aligned} \qquad (j = 1,2,\ldots,N) \qquad (1)$$

soient équivalentes à l'équation différentielle matricielle :

$$\frac{dL}{dt} = [M,L] \qquad (2)$$

L'intérêt de cette formulation réside dans le fait que les valeurs propres de la matrice L sont indépendantes du temps.

En effet, si on résoud le problème différentiel suivant :

$$\frac{dU}{dt} = MU$$

$$U(0) = I$$

l'équation *(2)* assure que :

$$\frac{d}{dt}(U^{-1}LU) = 0$$

et donc :

$$U^{-1}LU = L(0)$$

En conséquence, les valeurs propres de L sont invariantes au cours de l'évolution ainsi que les coefficients du polynôme caractéristique de L :

$$\Delta_n(\lambda) = \lambda^n + \lambda I_1^{\,n-1} + \ldots + I_n$$

Cette méthode permet ainsi d'obtenir des constantes du mouvement.

Pour prouver l'intégrabilité du système considéré, il faut, en plus, choisir la matrice L de manière à avoir un nombre suffisant de telles constantes, qui soient de plus indépendantes et en involution.

En fait, l'existence de paires de Lax est directement reliée à l'existence d'algèbres de Lie semi-simples (c'est-à-dire sommes directes d'algèbres de Lie simples). Cette remarque, compte-tenu du fait que l'on a une classification des algèbres de Lie simples de rang fini N (classification de Cartan), facilite grandement la recherche de paires de Lax [5].

En face d'un système hamiltonien de dimension N, on est donc amené à considérer les différentes algèbres de Lie simples de rang N (le rang de l'algèbre est choisi égal à la dimension du système de manière à obtenir suffisamment de constantes).

On choisit ensuite des matrices L et M dans une représentation matricielle de dimension minimale de l'algèbre (pour que les constantes correspondantes soient en involution), de manière à ce qu'elles vérifient l'équation *(2)*.

Le problème difficile qui se pose est en fait celui du choix simultané d'un bon système de coordonnées, et d'une bonne représentation (L,M) pour avoir équivalence entre les équations *(1)* et *(2)*. Ce choix peut être guidé par le fait que l'on a généralement : $Tr(L^2) = H$.

Les chaînes de Toda ou de Calogéro à N dimensions et conditions aux limites périodiques se prêtent bien à cette méthode. La structure respective de leurs hamiltoniens permet d'écrire facilement les équations de mouvement sous la forme de Lax.

$$\left(H = \frac{1}{2}\sum_j p_j^2 + \sum_{j<k} V(q_j - q_k) \text{ dans le cas de la chaîne de Calogéro,}\right.$$
$$\left.V(q) = \frac{1}{q^2}\right)$$

$$H = \frac{1}{2}\sum_j p_j^2 + \sum_j V(q_j - q_{j+1}) \text{ dans le cas de la chaîne de Toda,}$$
$$V(q) = e^{-q})$$

Dans la suite de cette partie, nous présentons une étude détaillée de la chaîne de Toda en faibles dimensions avec des masses et des portées d'interaction inégales. Nous nous limitons au cas d'une chaîne de trois particules à bout libre (qui se ramène à un système dynamique de dimension 2 si on introduit les coordonnées du centre de masse) et au cas de deux particules avec bouts fixes.

Nous prouvons l'intégrabilité des cas qui étaient des candidats d'après l'analyse Painlevé et ce, par deux méthodes différentes : le calcul direct des intégrales du mouvement et la recherche de paires de Lax associées au système.

Cette dernière méthode nous a permis non seulement d'identifier les cas intégrables mais aussi de calculer les constantes du mouvement de la manière qui a été indiquée précédemment.

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INTEGRALS OF MOTION FOR TODA SYSTEMS WITH UNEQUAL MASSES

B. DORIZZI

B. GRAMMATICOS

R. PADJEN

V. PAPAGEORGIOU

CNET/PAA/TIM/MTI

38-40, Avenue du Général Leclerc

92131 ISSY-LES-MOULINEAUX

. A paraître dans Journal of Mathematical Physics.

ABSTRACT

We present new integrals of motion for the Toda Lattice (chain of particles in one dimension with exponential interaction) for two special cases of boundary conditions : Free end Lattice with three non-equal-mass particles and fixed end Lattice for two particles. In both cases, we use two distinct approaches in order to identify the integrable cases : direct search of the integral of motion and group theoretical methods. Our results are on agreement with the predictions of Painlevé analysis.

I. - INTRODUCTION

The Toda Lattice [1] is a one-dimensional system of equal mass particles interacting via non linear forces : the interactions occur only between nearest neighbors and are of exponential type. With this lattice, we are in presence of a "small miracle" : The system is integrable for any number of particles in the chain. (Integrability, in the case of Hamiltonian systems of N degrees of freedom, is synonymous to existence of N analytical, single valued integrals of the motion, time independent and in involution). The integrability for periodic boundary conditions, has been shown independently by Hénon [2], Flaschka [3], and Manakov [4]. The first has explicitely calculated the integrals of the motion, while the two others deduced the integrability from group theoretical methods.

Moreover, this system is also integrable for other boundary conditions. The integrals in the case of fixed end lattice (two ends of the chain are set permanently equal to zero) can be easily deduced from the periodic case as shown by Hénon [2]. The free end lattice (the beginning of the chain is set to $-\infty$, the end is set to $+\infty$), has been discussed by Moser [5]. He has shown that the system admits the Lax pair representation and has used the latter to calculate the N integrals of the motion.

The aim of this work is to use this twofold approach, i.e. direct computation of constant of motion and group theoretical methods (i.e. search for Lax representations), in order to study other cases of integrability in low dimensional and unequal mass systems. Indeed, the great number of integrability conditions to be satisfied in the general N body case compells us to deal with particles of equal masses and equal ranges of interaction. However, one can reasonably hope that for systems of two or three particles,

the constraints of equal masses can be relaxed. A very useful tool for the identification of integrability candidates is the Painlevé Criterion as introduced by the work of Ablowitz - Ramani - Segur [6]. They have conjectured that integrability is intimately related to the analytic properties of the solutions of the equations of motion. Namely, whenever the solutions possess the Painlevé property, i.e. their only movable singularities on the complex-time plane are poles, the system is integrable. The reciprocal is also true and has been verified for the known integrable systems. However recent results of Ramani, Dorizzi and Grammaticos [7] have shown that for two-dimensional hamiltonian systems, integrability can sometimes be associated to some weakened Painlevé property. In the case of the Toda system, at hand such a generalization is unnecessary, and the known integrable cases possess the full Painlevé property.

In this paper, we will concentrate on two particular forms of the Toda system :

1°) The free-end Lattice with three masses :

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2} + e^{\varepsilon (q_1 - q_2)} + e^{q_2 - q_3}$$

The integrability of this system cannot be verified numerically by the surfaces of section method since the above hamiltonian describes a scattering problem.

In a recent publication [8], Bountis, Segur and Vivaldi have found that the Painlevé property is satisfied for three values of the parameters :

a) $m_1 = \frac{\varepsilon (2\varepsilon - 1)}{2 - \varepsilon}$, $m_2 = 2\varepsilon - 1$, $\frac{1}{2} < \varepsilon < 2$

b) $m_1 = \frac{\varepsilon (\varepsilon - 1)}{2 - \varepsilon}$, $m_2 = \varepsilon - 1$, $1 < \varepsilon < 2$

c) $m_1 = \frac{3\varepsilon (2\varepsilon - 1)}{2 - 3\varepsilon}$, $m_2 = 2\varepsilon - 1$, $\frac{1}{2} < \varepsilon < \frac{2}{3}$

The integrability of case a) has been proved rigourously by Moser [5] and Bogoyavlenski [9]. Cases b) and c) can be deduced from the theorem of Bogoyavlenski [9] and are explicitely studied in [10].

In Section II, we will present a direct computation of the constants of motion for all the three cases a), b) and c) above.

2°) The fixed end lattice with two masses :

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + e^{-\delta q_1} + e^{\varepsilon (q_1 - q_2)} + e^{q_2}$$

Casati and Ford predicted integrability for $\varepsilon = \delta = 1$, $m_1/m_2 = 1$, based on a numerical study.

The Painlevé analysis of Bountis and al suggests three cases of integrability :

a) $m_1/m_2 = 1$, $\delta = \varepsilon = 1$

b) $m_1/m_2 = 1$, $\delta = 1$, $\varepsilon = 1/2$

c) $m_1/m_2 = 1/3$, $\delta = 1$, $\varepsilon = 1/2$.

On the other hand, the Lie algebra study of Bogoyavlensky yields two integrability candidates : case b) above and a new one :

d) $m_1/m_2 = 1/3$, $\delta = 1/3$, $\varepsilon = 1/2$.

In fact, we find five integrable combinations of masses and ranges : cases a), b), c), d) listed above and

e) $m_1/m_2 = 1$, $\delta = 1/2$, $\varepsilon = 1/2$.

Actually all five integrable cases can be predicted by the Painleve analysis, as was shown by A. Ramani [11].

Section IV deals exclusively with group theoretical methods. This approach leads to the identification of the integrable cases listed above as can be shown in several recent works [9, 10, 12]. However their approaches are general and rather abstract. Our aim in this section is to explicit these results in the various cases of integrability. In particular we exhibit the Cartan matrix for the three different cases for which the free end Toda lattice admits the Lax pair representation. This allows the calculation of the integrals in each case. The same procedure is applied to the fixed end case.

II. - FREE END LATTICE

Let us consider a free end lattice with three, non-equal-mass, particles. In that case, the Hamiltonian governing the system reads :

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + e^{\varepsilon (q_1 - q_2)} + e^{q_2 - q_3} \qquad (2-1)$$

In order to explicit the motion of the center of mass z of the system, we introduce the following change of variable

$$\begin{cases} x = \varepsilon (q_1 - q_2) \\ y = q_2 - q_3 \\ z = m_1 q_1 + m_2 q_2 + m_3 q_3 \end{cases} \qquad (2-2)$$

The equations of motion associated to the system are derived from the Lagrange equations :

$$\begin{cases} \frac{\partial}{\partial t} (m_1 \dot{q}_1) = - \varepsilon X \\ \frac{\partial}{\partial t} (m_2 \dot{q}_2) = \varepsilon X - Y \\ \frac{\partial}{\partial t} (m_3 \dot{q}_3) = Y \end{cases} \qquad (2-3)$$

$$\text{with } X = e^{\varepsilon (q_1 - q_2)} = e^x$$

$$Y = e^{q_2 - q_3} = e^y$$

From equations (1 - 2), it is obvious that

$$\ddot{z} = m_1 \ddot{q}_1 + m_2 \ddot{q}_2 + m_3 \ddot{q}_3 = 0. \qquad (2-4)$$

One can also obtain, from (1 - 2), the equations of the motion of x and y :

$$\begin{cases} \ddot{x} = \varepsilon\,(\ddot{q}_1 - \ddot{q}_2) = \dfrac{\varepsilon}{m_2}\left(Y - \varepsilon\,\dfrac{m_1 + m_2}{m_1}\,X\right) \\ \ddot{y} = \ddot{q}_2 - \ddot{q}_3 = \dfrac{\varepsilon}{m_2}\left(X - \dfrac{m_2 + m_3}{\varepsilon\, m_3}\,Y\right) \end{cases} \qquad (2-5)$$

which, after a scaling in time, read :

$$\begin{cases} \ddot{x} = Y - \alpha\, X \\ \ddot{y} = X - \beta\, Y \end{cases} \qquad (2-6)$$

$$\text{with } \alpha = \frac{\varepsilon\,(m_1 + m_2)}{m_1}, \quad \beta = \frac{m_2 + m_3}{\varepsilon\, m_3}$$

The equations for (x, y) and z are separated. So, for the system (2 - 3) to be completely integrable it is sufficient that the system (2 - 6) itself, be integrable. It already possesses a first integral of motion, namely the energy, obtained by substraction of the center of mass energy from the hamiltonian H (written in terms of x and y)

$$E = \frac{m_3\,(m_1 + m_2)}{2\,\alpha\,(m_1 + m_2 + m_3)}\,(\beta\dot{x}^2 + 2\dot{x}\dot{y} + \alpha\dot{y}^2) + X + Y. \qquad (2-7)$$

We will then look for a second integral polynomial in $\dot{x}$ and $\dot{y}$, i.e. of the form :

$$C = \sum_{n=0}^{N} \sum_{k=0}^{n} f_k^n\, \dot{x}^{n-k}\, \dot{y}^k \qquad (2-8)$$

where f_k^n are general functions of x, y.

(Only powers of the same parity in the velocities will appear in the sum due to the time reversal invariance of the Hamiltonian). We will exhaust all the possible constants of the form (2 - 8) up to order N = 6.

This method has already proved to be a valuable tool in the study of integrable dynamical systems with polynomial potentials. In particular, in the case of polynomial potentials of degree three, it allows the calculation

of the integrals of motion in all the Painlevé cases [13]. This will be the case for the Toda Lattice as well.

The complete details for the search of a constant of order 2, 3 and 4 with a general potential are exposed in ref [13]. In this paper, we will just present the calculation in the particular case of the Toda potentials.

A calculation at order 2 does not give any result.

At order 3, the form of the integral is :

$$C = f_0 \dot{x}^3 + f_1 \dot{x}^2 \dot{y} + f_2 \dot{x} \dot{y}^2 + f_3 \dot{y}^3 + g_0 \dot{x} + g_1 \dot{y} \qquad (2-9)$$

A direct computation shows that the f_i's must be constant.

The condition $\frac{dC}{dt} = 0$ leads to a system of partial differential equations for the g_i's :

$$\begin{cases} 3f_o \ddot{x} + f_1 \ddot{y} + \dfrac{\partial g_o}{\partial x} = 0 \\ 2 f_1 \ddot{x} + 2f_2 \ddot{y} + \dfrac{\partial g_o}{\partial y} + \dfrac{\partial g_1}{\partial x} = 0 \\ f_2 \ddot{x} + 3f_3 \ddot{y} + \dfrac{\partial g_1}{\partial y} = 0 \end{cases}$$

The compatibility condition, necessary for the integration of the equations for the g_i's :

$$\frac{\partial^2}{\partial x^2} (f_2 \ddot{x} + 3f_3 \ddot{y}) - \frac{\partial}{\partial x \partial y} (2f_1 \ddot{x} + 2f_2 \ddot{y}) + \frac{\partial^2}{\partial y^2} (3f_0 \ddot{x} + f_1 \ddot{y}) = 0, \qquad (2-10)$$

is here considerably simplified thanks to the form of the equations (2 - 6) for $\ddot{x}$ and $\ddot{y}$. Indeed, as the functions X and Y depend only on x and y respectively, we are led to the conditions :

$$\begin{cases} 3f_0 = \beta f_1 \\ 3f_3 = \alpha f_2 \end{cases} \qquad (2-11)$$

One can then easily integrate the equations for the g_i's

$$\begin{cases} g_0 = (\alpha\beta - 1) f_1 X + \frac{2}{3} (3f_2 - f_1) Y \\ g_1 = 2 (\alpha f_1 - f_2) X + (\alpha\beta - 1) f_2 Y \end{cases} \qquad (2-12)$$

It remains just to satisfy the condition :

$$g_0 \ddot{x} + g_1 \ddot{y} = 0.$$

This expression is an identity in terms of the independent functions of x and y : X^2, XY, Y^2. Thus the coefficients f_1 and f_2 must satisfy three linea equations :

$$\begin{cases} \alpha f_1 (3 - \alpha\beta) - 2f_2 = 0 \\ -2f_1 + \beta f_2 (3 - \alpha\beta) = 0 \\ f_1 (2\alpha - \alpha\beta - 1) + f_2 (2\beta - \alpha\beta - 1) = 0 \end{cases} \qquad (2-13)$$

This system has a non trivial solution (f_1, f_2) whenever α and β satisfy the two equations :

$$\begin{cases} \alpha (3 - \alpha\beta) (2\beta - \alpha\beta - 1) + 2 (2\alpha - \alpha\beta - 1) = 0 \\ \beta (3 - \alpha\beta) (2\alpha - \alpha\beta - 1) + 2 (2\beta - \alpha\beta - 1) = 0 , \end{cases} \qquad (2-14)$$

or equivalently :

$$(\alpha - \beta) (\alpha\beta - 1)^2 = 0$$

The condition $\alpha\beta = 1$ is to be discarded because, in this case, the change of variable (2 - 2) is not defined.

Thus $\alpha = \beta$ $(\neq 1)$,

the first equation (2 - 14) reads then :

$$(\alpha - 1)^2 (\alpha + 1)^2 (\alpha - 2) = 0.$$

When $\alpha = \beta = 2$ $(f_2 = - f_1)$ or, in term of ε :

$$m_1 = \frac{\varepsilon (2\varepsilon - 1)}{2 - \varepsilon}, \quad m_2 = 2\varepsilon - 1, \quad m_3 = 1, \quad \frac{1}{2} < \varepsilon < 2,$$

the system (1 - 5) is integrable and admits apart from the energy a second constant of motion cubic in the velocities :

$$C_1 = 2 \dot{x}^3 + 3 \dot{x}^2 \dot{y} - 3 \dot{x} \dot{y}^2 - 2 \dot{y}^3 + 9 (e^x - 2e^y) \dot{x} + 9 (2 e^x - e^y) \dot{y} \qquad (2-15)$$

The associated free end Toda Lattice is then also integrable. We recover thus the first case quoted by Bountis et al. and treated independently by Moser and Bogoyavlenski.

Let us consider now an integral of order four in the velocities :

$$C = f_0 \dot{x}^4 + f_1 \dot{x}^3 \dot{y} + f_2 \dot{x}^2 \dot{y}^2 + f_3 \dot{x} \dot{y}^3 + f_4 \dot{y}^4 + g_0 \dot{x}^2 + g_1 \dot{x} \dot{y} + g_2 \dot{y}^2 + h(x, y) \qquad (2-16)$$

As in the preceeding case, we can restrict ourselves to constant f_i's and f_4 can be taken equal to zero by adding to C the suitable multiple of the square of the Hamiltonian.

We recall the form of the compatibility condition for the integrability of the g_i's :

$$\frac{\partial^3}{\partial y^3}(4f_0 \ddot{x} + f_1 \ddot{y}) - \frac{\partial^3}{\partial y^2 \partial x}(3f_1 \ddot{x} + 2f_2 \ddot{y}) + \frac{\partial^3}{\partial y \partial x^2}(2f_2 \ddot{x} + 3f_3 \ddot{y}) - \frac{\partial^3}{\partial x^3}(f_3 \ddot{x} + 4f_4 \ddot{y}) = 0 \qquad (2-17)$$

In the particular case where $\ddot{x}$ and $\ddot{y}$ are given by equation (1 - 5), this last equation reduces to :

$$\frac{\partial^3}{\partial y^3}(4f_0 Y - \beta f_1 Y) + \frac{\partial^3}{\partial x^3}(f_3 \alpha X) = 0 ;$$

thus

$$\begin{cases} f_3 = 0 \\ 4f_0 = \beta f_1 \end{cases} \qquad (2-18)$$

Using (2 - 17), the equations for the g_i's can be integrated to give :

$$\begin{cases} g_0 = f_1 (\alpha\beta - 1) X + (2\beta f_2 - 3f_1) Y \\ g_1 = (3\alpha f_1 - 2f_2) X - 2f_2 Y \\ g_2 = 2\alpha f_2 X \end{cases} \qquad (2-19)$$

The second compatibility condition for the integrability of h :

$$\frac{\partial}{\partial y} (2g_0 \ddot{x} + g_1 \ddot{y}) = \frac{\partial}{\partial x} (g_1 \ddot{x} + 2g_2 \ddot{y}) \qquad (2-20)$$

is an identity in terms of the independent functions X^2, XY, Y^2, as in the preceeding case. We obtain thus a system in terms of f_1, f_2, α, β :

$$\begin{cases} \alpha f_1 - 2f_2 = 0 \\ f_1 - \beta f_2 = 0 \\ f_1 (3\alpha - \alpha\beta - 2) + 2f_2 (\beta - \alpha) = 0 \end{cases} \qquad (2-21)$$

If the two following conditions hold :

$$\alpha\beta = 2$$

$$(1 - \alpha) (\alpha - 2) = 0,$$

the system (2-21) will have a non trivial solution (f_1, f_2). We have thus obtained that, whenever

$$\text{(a)} \quad \alpha = 1, \ \beta = 2 \qquad \text{or} \quad \text{(b)} \quad \alpha = 2, \ \beta = 1 \qquad (2-22)$$

the system (2-6) is integrable and possesses a second integral quartic in the velocities.

The two cases (a) and (b) are related by change ε in 2ε, i.e. a sclaling on x (x in $\frac{x}{2}$). The case (b) writes, in term of m_1, m_2, m_3 ; ε :

$$m_1 = \frac{\varepsilon (\varepsilon - 1)}{2 - \varepsilon}, \ m_2 = \varepsilon - 1, \ m_3 = 1, \quad 1 < \varepsilon < 2, \qquad (2-23)$$

In that case, the constant C_2 is given by :

$$C_2 = \dot{x}^4 + 4\,\dot{x}^3\,\dot{y} + 4\,\dot{x}^2\,\dot{y}^2 + 4\,(e^x - e^y)\,\dot{x}^2 + 8\,(2e^x - e^y)\,\dot{x}\,\dot{y} + 16\,e^x\,\dot{y}^2 + 4\,e^{2y} \qquad (2-24)$$

So the Toda Lattice related to that case is integrable. The values of the parameters m_i et ε correspond to the second case provided by the Painlevé Analysis (cf. Ref [8]).

The case of a constant of motion quintic in the velocities has been examined but does not yield any positive result.

Let us now consider the case of a constant of order six in the velocities. The computations are similar but more complicated.

The form of a sixth order constant is :

$$\begin{aligned} C = {} & e_0\,\dot{x}^6 + e_1\,\dot{x}^5\,\dot{y} + e_2\,\dot{x}^4\,\dot{y}^2 + e_3\,\dot{x}^3\,\dot{y}^3 + e_4\,\dot{x}^2\,\dot{y}^4 + e_5\,\dot{x}\,\dot{y}^5 + e_6\,\dot{y}^6 \\ & + f_0\,\dot{x}^4 + f_1\,\dot{x}^3\,\dot{y} + f_2\,\dot{x}^2\,\dot{y}^2 + f_3\,\dot{x}\,\dot{y}^3 + f_4\,\dot{y}^4 \\ & + g_0\,\dot{x}^2 + g_1\,\dot{x}\,\dot{y} + g_2\,\dot{y}^2 + h \end{aligned} \qquad (2-25)$$

As in the preceeding cases, the choice of constant e_i's emerges naturally in the computation (e_6 can be taken equal to zero by adding to C a multiple of the cube of the Hamiltonian). Now, in order to equate to zero the coefficients of order five in $\frac{dC}{dt} = 0$, we obtain a system of partial differential Equations for the f_i, which leads to the new compatibility condition (2 - 26). As soon as this last condition is satisfied, one can calculate the functions f_i's . The problem is then reduced to the search of the g_i's and h from relations that read exactly the same as in the previous case of constants of order 4 in the velocities.

The new compatibility condition writes :

$$\frac{\partial^5}{\partial y^5}(6e_0\,\ddot{x} + e_1\,\ddot{y}) - \frac{\partial^5}{\partial y^4\,\partial x}(5e_1\,\ddot{x} + 2e_2\,\ddot{y}) + \frac{\partial^5}{\partial x^2\,\partial y^3}(4e_2\,\ddot{x} + 3e_3\,\ddot{y})$$

$$- \frac{\partial^5}{\partial x^3\,\partial y^2}(3e_3\,\ddot{x} + 4e_4\,\ddot{y}) + \frac{\partial^5}{\partial x^4\,\partial y}(2e_4\,\ddot{x} + 5e_5\,\ddot{y}) - \frac{\partial^5}{\partial x^5}(e_5\,\ddot{x}) = 0 \qquad (2-26)$$

which immediately gives :

$$\begin{cases} 6e_0 = e_1\,\beta \\ e_5 = 0 \end{cases} \qquad (2-27)$$

The integration for the f_i is straight forward :

$$\begin{cases} f_0 = e_1(\alpha\beta - 1)X + (2e_2\,\beta - 5e_1)Y \equiv A_0 X + B_0 Y \\ f_1 = (5e_1\,\alpha - 2e_2)X + (3e_3\,\beta - 4e_2)Y \equiv A_1 X + B_1 Y \\ f_2 = (4e_2\,\alpha - 3e_3)X + (4e_4\,\beta - 3e_3)Y \equiv A_2 X + B_2 Y \\ f_3 = (3e_3\,\alpha - 4e_4)X - 2e_4 Y \equiv A_3 X + B_3 Y \\ f_4 = 2e_4\,\alpha\,X \equiv A_4 X \end{cases} \qquad (2-28)$$

The next compatibility condition (2 - 17) reduces to :

$$\begin{cases} \beta B_1 - 4B_0 = 0 \\ \alpha A_3 - 4A_4 = 0 \\ 2(B_2\,\alpha - A_2) + 3(A_3\,\alpha - B_3) + 4(B_0\,\alpha - A_0) + (A_1\,\alpha - B_1) = \\ 2(A_2\,\beta - B_2) + 3(B_1\,\alpha - A_1) + 4(A_4\,\beta - B_4) + (B_3\,\alpha - A_3) \end{cases} \qquad (2-29)$$

If α, β, and the constants e_i satisfy eq. (2 - 28), it is then possible to calculate the functions g_i :

$$g_0 = \frac{1}{2}(4A_0\alpha - A_1)X^2 + \left(4(B_0\alpha - A_0) + (A_1\beta - B_1)\right)XY + \frac{1}{2}\left(2(B_2\beta - B_1)\right)Y^2$$
$$\equiv C_0 X^2 + D_0 XY + E_0 Y^2$$

$$g_1 = \frac{1}{2}(3A_1\alpha - 2A_2)X^2 + \left[3(B_1\alpha - A_1) + 2(A_2\beta - B_2) - 4(B_0\alpha - A_0) - (A_1\beta - B_1)\right]XY + \frac{1}{2}(3B_3\beta - 2B_2)Y^2 \equiv G X^2 + D_1 XY + E_1 Y^2$$

$$g_2 = \frac{1}{2}(2A_2\alpha - 3A_3)X^2 + \left[(B_3\alpha - A_3) + 4(A_4\beta - B_4)\right]XY + \frac{1}{2}(4B_4\mu - B_3)Y^2$$
$$\equiv C_2 X^2 + D_2 XY + E_2 Y^2 \qquad (2-30)$$

C_i, D_i, E_i ; i = 0, 1, 2 are complicated expressions in terms of α, β and A_j, B_j, j = 0, 1, 2, 3, 4.

The last compatibility relation (2 - 20) will give four other constraints :

$$\begin{cases} \alpha C_1 - 2C_2 = 0 \\ 2(D_0\alpha - C_0) + (\beta C_1 - D_1) - 2(\alpha D_1 - C_1) - 4(C_2\beta - D_2) = 0 \\ 2(D_2\beta - E_2) + (\alpha E_1 - D_1) - 2(\beta D_1 - E_1) - 4(\alpha E_0 - D_0) = 0 \\ \beta E_1 - 2E_0 = 0 \end{cases} \qquad (2-31)$$

The 9 equations 1 - 26, 1 - 28, 1 - 30 summarize in terms of α, β and the e_i's the conditions for which the system possesses an integral of order six in the velocities. They form a linear, homogeneous system of 9 equations for the six unknown e_i. It is possible to show that, in order to get a non trivial solution to this system, α and β have to take the following values (up to permutations of α and β).

(a) $\alpha = 2$, $\beta = 2$: the integral is the square of the constant C_1 of degree three found previously : (2 - 15)

(b) $\alpha = 2$, $\beta = 1$: the integral is the product of the constant C_2 (2 - 24) of degree four, found previously, with the Hamiltonian.

(c) $\alpha = \frac{2}{3}$, $\beta = 2$ this is a new case which corresponds to :

$$m_1 = \frac{3\varepsilon\ (2\varepsilon - 1)}{2 - 3\varepsilon}\ ,\quad m_2 = 2\varepsilon - 1,\quad m_3 = 1,\quad \frac{1}{2} < \varepsilon < \frac{2}{3} \qquad (2 - 32)$$

For these values of the parameters, Bountis and al. found the system to be Painlevé. It is, in fact, integrable with a non degenerate constant of the form :

$$\begin{aligned} C = {} & 4\,\dot{x}^6 + 12\,\dot{x}^5\,\dot{y} + 13\,\dot{x}^4\,\dot{y}^2 + 6\,\dot{x}^3\,\dot{y}^3 + \dot{x}^2\,\dot{y}^4 + 4\,(e^x - 2e^y)\,\dot{x}^4 \\ & + (14e^x - 16e^y)\,\dot{x}^3\,\dot{y} + 10\left(\frac{5}{3}\,e^x - e^y\right)\dot{x}^2\,\dot{y}^2 + 2\,(4e^x - e^y)\,\dot{x}\,\dot{y}^3 \\ & + \frac{4}{3}\,e^x\,\dot{y}^4 + \left(-\frac{5}{3}\,e^{2x} + \frac{20}{3}\,e^{x+y} + 4e^{2y}\right)\dot{y}^2 \\ & + \left(-\frac{8}{3}\,e^{2x} + 6e^{x+y} + 4e^{2y}\right)\dot{x}\,\dot{y} + \left(-\frac{8}{9}\,e^{2x} + \frac{4}{3}\,e^{x+y} + e^{2y}\right)\dot{y}^2 \\ & + \frac{4}{27}\,e^{3x} + \frac{44}{9}\,e^{2x+y}. \end{aligned} \qquad (2 - 33)$$

So, every case of integrability predicted by the Painlevé analysis [8] was indeed recovered by the direct approach for the calculation of the integrals of motion.

III. - THE FIXED - END LATTICE FOR TWO PARTICLES :

Let us consider a fixed end lattice with two non equal masses and non equal interactions. The form of the Hamiltonian governing the system is then :

$$H = \frac{p_x^2}{2m_1} + \frac{p_y^2}{2m_2} + e^{-\delta x} + e^{\varepsilon (x - y)} + e^{y} \qquad (3-1)$$

In order to alleviate the notations, we put :

$$\begin{aligned} X &= e^{-\delta x} \\ D &= e^{\varepsilon(x - y)} \\ Y &= e^{y} \end{aligned} \qquad (3-2)$$

The equations of motion read (up to a scaling in time) :

$$\begin{cases} \ddot{x} = \delta X - \varepsilon D \\ \ddot{y} = a\,(\varepsilon D - Y) \end{cases} \qquad (3-3)$$

$$a = \frac{m_1}{m_2}$$

At this point, we remark that a change :

$a' = \frac{1}{a}$, $\delta' = \frac{1}{\delta}$, $\varepsilon' = \frac{\varepsilon}{\delta}$ leads to the same form of the equations of motion for $\xi = -\delta y$, $\eta = -\delta x$. These cases are then equivalent up to an x, y permutation and a scaling.

As in the preceeding section, we will systematicaly look for a second constant of motion polynomial in the velocities. We will not burden the presentation by exposing the computations at order 2 and 3 ; they did not yield any positive result.

Let us then begin with a quartic constant (form (2 - 15)). As previously, it is sufficient to deal with constant coefficients f_i ($f_4 = 0$).

The first compatibility condition (2 - 17) reduces to :

$$f_1 = f_3 = 0 \qquad (3 - 4a)$$

$$f_2 (1 - a) + 2f_0 = 0 \qquad (3 - 4b)$$

To obtain this result we use again explicitely the fact that the functions X and Y depend only respectively on x and y.

We integrate and find the functions g_i :

$$\begin{aligned} g_0 &= -4f_0 (X + D) - 2a f_2 Y \equiv F_0 (X + D) + F_2 Y \\ g_1 &= -4f_0 D + 2a f_2 D \equiv (F_0 - F_2) D \\ g_2 &= -2f_2 X \equiv \left(\frac{F_2}{a}\right) X \end{aligned} \qquad (3 - 5)$$

Once the relations (3-4a, 3-4b) are fulfilled, it suffices to satisfy th second compatibility condition (2 - 20), for the system to possess an integr quartic in velocities. In this relation appear the independent functions of (x, y) XD, XY, DY, D^2. Thus (F_0, F_2) must be a non trivial solution of the

and :
$$\begin{cases} (1 - a) F_2 + a F_0 = 0 \quad \text{transcription of (3 - 4b)} \\ (1 - \varepsilon) (a F_0 - F_2) (a - 2\varepsilon)) = 0 \\ (3 - a) F_0 + (a - 1) F_2 = 0 \\ (\delta^2 + 2\varepsilon^2 - 3\delta\varepsilon) (F_0 - F_2) = 0 \end{cases} \qquad (3 - 6)$$

(the coefficient of XY is always zero)

It is obvious that the solutions are :

$$F_0 = 0, \quad F_2 \neq 0 \text{ and}$$

$$a = 1, \quad \varepsilon = 1 \text{ or } \varepsilon = \frac{1}{2} \text{ and}$$

$$\frac{\delta}{\varepsilon} = 2 \text{ or } \frac{\delta}{\varepsilon} = 1.$$

We finaly find three distinct cases for which a quartic integral exists :

a) $\frac{m_1}{m_2} = 1, \quad \delta = \varepsilon = 1$ (3 - 7)

b) $\frac{m_1}{m_2} = 1, \quad \delta = 1, \ \varepsilon = \frac{1}{2}$ (3 - 8)

e) $\frac{m_1}{m_2} = 1, \quad \delta = \frac{1}{2}, \ \varepsilon = \frac{1}{2}$ (equivalent to $\delta = 2, \ \varepsilon = 1$). (3 - 9)

(The classification a), b), e) is that of introduction).

The explicit values of the constants are :

a) $C = \frac{\dot{x}^2 \dot{y}^2}{2} + e^y \dot{x}^2 - e^{x-y} \dot{x} \dot{y} + e^{-x} \dot{y}^2 + \frac{e^{2(x-y)}}{2} + e^x + 2e^{y-x} + e^{-y}.$

(3 - 10)

b) $C = \frac{\dot{x}^2 \dot{y}^2}{2} + e^y \dot{x}^2 - e^{(x-y)/2} \dot{x} \dot{y} + e^{-x} \dot{y}^2 - \frac{e^{x-y}}{2} + 2 e^{y-x}$ (3 - 11)

e) $C = \frac{\dot{x}^2 \dot{y}^2}{2} + e^y \dot{x}^2 - e^{(x-y)/2} \dot{x} \dot{y} + e^{-\frac{x}{2}} \dot{y}^2 + \frac{e^{x-y}}{2} + 2 e^{y-\frac{x}{2}} + e^{-\frac{y}{2}}$

(3 - 12)

In order to find the integrals for the other candidates of integrability (cases c) and d) of introduction), it is necessary, as in the preceeding section, to perform the calculations at order six. (Recall $e_6 = 0$).

The first compatibility condition (1 - 25) easily gives a first relation between the e_i's :

$$\begin{cases} e_1 = e_5 = 0 \\ E_0 + (2 - a) E_2 + (1 - a) E_3 + (1 - 2a) E_4 = 0 \\ (E_0 = 6e_0, \ E_2 = 2e_2, \ E_3 = 3e_3, \ E_4 = 2e_4) \end{cases} \qquad (3 - 13)$$

The integration of the equations for the f_i's is then straightforward :

$$\begin{cases} f_0 = E_0 X + a E_2 Y + E_0 D \\ f_1 = a E_3 Y + (E_0 - a E_2) D \\ f_2 = 2 E_2 X + (E_0 + (2 - a) E_2 - a E_3) D + 2 a E_4 Y \\ f_3 = E_3 X - E_4 D \\ f_4 = E_4 X \end{cases} \qquad (3-14)$$

The second compatibility condition (2 - 17) gives relations on the coefficient E_i :

$$\begin{cases} 3 E_0 (a - 3) + E_2 (9a - 4 - 3a^2) + E_4 (1 - 3a) = 0 \qquad (3-15) \\ (\varepsilon - 1) \left(E_0 (1 - \varepsilon) (1 - 3\varepsilon) + E_2 (1 - \varepsilon) (3a\varepsilon - a - 4\varepsilon^2) + \varepsilon^2 E_4 (4a - 4a\varepsilon - 3 + 4\varepsilon)\right) \end{cases}$$

and : $$\left[E_0 \varepsilon (5\varepsilon\delta - 2\delta^2 - 4\varepsilon^2) + a\varepsilon E_2 (-5\varepsilon\delta a + 4\varepsilon^2 a + 2a\delta^2 - 4(\delta - \varepsilon)^2) + a(\varepsilon - \delta)^2 E_4 (4\varepsilon a - \delta)\right](\varepsilon - \delta) = 0 \qquad (3-16)$$

This system (3 - 13, 3 - 15, 3 - 16) is satisfied for :

$$\begin{cases} a = 1, \quad \varepsilon = 1, \quad \delta = 1 \\ a = 1, \quad \varepsilon = \frac{1}{2}, \quad \delta = \frac{1}{2} \\ a = 1, \quad \varepsilon = 1, \quad \delta = 2 \end{cases}$$

which correspond to the preceeding cases (3-7, 3-8, 3-9).

For $a = \frac{1}{3}$, $\varepsilon = \frac{1}{2}$, equations (3 - 13, 3 - 15) are satisfied and yield

$$\begin{cases} E_2 = -6E_0 \\ E_4 = 27E_0 \end{cases}$$

In order to satisfy equation (3 - 16) for these values of a and ε, δ must take one of the following values :

$$\delta = 1, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{1}{2}.$$

Now, once the system (3 - 13, 3 - 15, 3 - 16) is satisfied, the explicit calculation of the g_i's reads :

$$\begin{aligned} g_0 &= 2E_0 X^2 + 4E_0 XD + 4a E_2 XY + \frac{YD}{\varepsilon}(4a E_2 \varepsilon - a^2 E_2 + a E_0) \\ &\quad + \frac{D^2}{2}(4E_0 - a E_0 + a^2 E_2) + 2a^2 E_4 Y^2 \\ g_1 &= \frac{D^2}{2}(7E_0 - 3a E_0 - 7a E_2 + 3a^2 E_2) + \frac{3\delta - 4\varepsilon}{\varepsilon - \delta} XD (a E_2 - E_0) \\ &\quad + \frac{YD}{\varepsilon}(- 4\varepsilon a^2 E_4 + 3a E_0 - 3a^2 E_2 + 4a \varepsilon E_2 - \frac{a E_0}{\varepsilon} + \frac{a^2 E_2}{\varepsilon}) \\ g_2 &= \frac{E_4}{2} D^2 - \frac{XD}{\varepsilon}(E_4 \delta - 4a \varepsilon E_4) + 4 a E_4 XY + 2 E_2 X^2. \end{aligned} \tag{3 - 17}$$

At this point, there remains a last relation (2 - 20) to ensure for the existence of the function h of (2 - 25) and thus the existence of integral at this order. We can now explicit the values of g_i in (3 - 17) for particular cases of values of a, ε and δ solutions of (3 - 13, 3 - 15, 3 - 16) and check whether relation (2 - 20) holds or not.

1°) $a = \frac{1}{3}$, $\varepsilon = \frac{1}{2}$, $\delta = 1$. (Case c) in the introduction)

$$\begin{cases} g_0 = \frac{2 X^2}{3} + \frac{4 XD}{3} - \frac{8 XD}{3} - 2YD + \frac{D^2}{2} + 2 Y^2 \\ g_1 = 3 D^2 + 2 XD - 6 YD \\ g_2 = \frac{9 D^2}{2} - 6 XD + 12 XY - 4 X^2 \end{cases} \tag{3 - 18}$$

(with $3 E_0 = 1$)

Relation (1 - 19) is verified and leads to the following value of the constant :

$$C = \frac{\dot{x}^6}{18} - 2\,\dot{x}^4\,\dot{y}^2 + 9\,\dot{x}^2\,\dot{y}^4 + (e^{-x} - 2\,e^{y} + e^{\frac{(x-y)}{2}})\,\frac{\dot{x}^4}{3}$$

$$+ e^{\frac{(x-y)}{2}}\,\dot{x}^3\,\dot{y} + (6\,e^{y} - 3e^{\frac{(x-y)}{2}} - 4\,e^{-x})\,\dot{x}^2\,\dot{y}^2 - 9\,e^{\frac{(x-y)}{2}}\,\dot{x}\,\dot{y}^3$$

$$+ 9\,e^{-x}\,\dot{y}^4 + (2\,e^{-2x} + 4\,e^{\frac{-(x+y)}{2}} - 8\,e^{y-x} - 6\,e^{\frac{(x+y)}{2}}$$

$$+\frac{3}{2}\,e^{x-y} + 6\,e^{2y})\,\frac{\dot{x}^2}{3} + (3\,e^{x-y} + 2\,e^{\frac{-(x+y)}{2}} - 6\,e^{\frac{(x+y)}{2}})\,\dot{x}\,\dot{y}$$

$$+\frac{9}{2}\,e^{x-y} - 6\,e^{\frac{-(x+y)}{2}} + 12\,e^{y-x} - 4\,e^{-2x})\,\dot{y}^2 - 12\,e^{\frac{(y-x)}{2}}$$

$$+ 4\,e^{\frac{-(y+3x)}{2}} - 8\,e^{y-2x} - \frac{4}{3}\,e^{-3x} + 12\,e^{2y-x}\,. \qquad (3-19)$$

2°) $a = \frac{1}{3}$, $\varepsilon = \frac{1}{2}$, $\delta = \frac{1}{3}$ (Case d))

$$\begin{cases} g_0 = \frac{2\,X^2}{3} + \frac{4\,XD}{3} - \frac{8\,XY}{3} - 2\,YD + \frac{D^2}{2} + 2\,Y^2 \\ g_1 = 3\,D^2 + 6\,XD - 6\,YD \\ g_2 = \frac{9\,D^2}{2} + 6\,XD + 12\,YX - 4\,X^2 \end{cases} \qquad (3-20)$$

In this case also, one can easily check that relation (2 - 20) is verified and then compute the constant :

$$C = \frac{\dot{x}^6}{18} - 2\dot{x}^4\dot{y}^2 + 9\dot{x}^2\dot{y}^4 + (e^{\frac{-x}{3}} - 2e^{y} + e^{\frac{(x-y)}{2}})\frac{\dot{x}^4}{3}$$

$$+ e^{\frac{(x-y)}{2}}\dot{x}^3\dot{y} + (6e^{y} - 3e^{\frac{(x-y)}{2}} - 4e^{\frac{-x}{3}})\dot{x}^2\dot{y}^2 - 9e^{\frac{(x-y)}{2}}\dot{x}\dot{y}^3$$

$$+ 9e^{\frac{-x}{3}}\dot{y}^4 + (2e^{\frac{-2x}{3}} + 4e^{\frac{x}{6}-\frac{y}{2}} - 8e^{y-\frac{x}{3}} - 6e^{\frac{(x+y)}{2}}$$

$$+\frac{3}{2}e^{x-y} + 6e^{2y})\frac{\dot{x}^2}{3} + (3e^{x-y} + 6e^{\frac{x}{6}-\frac{y}{2}} - 6e^{\frac{(x+y)}{2}})\dot{x}\dot{y}$$

$$+\frac{9}{2}e^{x-y} + 6e^{\frac{x}{6}-\frac{y}{2}} + 12e^{y-\frac{x}{3}} - 4e^{\frac{-2x}{3}})\dot{y}^2 + e^{x}$$

$$-\frac{4}{9}e^{-x} - 4e^{2y-\frac{x}{3}} - 4e^{\frac{x}{6}+\frac{y}{2}} + \frac{(8e^{\frac{-2x}{3}+y})}{3}$$

$$-(\frac{4}{3})e^{-\frac{x}{6}-\frac{y}{2}}. \qquad (3-21)$$

3°) A precise analysis of the values :

$$a = \frac{1}{3}, \quad \varepsilon = \frac{1}{2}, \quad \delta = \frac{2}{3} \qquad \text{and}$$

$$a = \frac{1}{3}, \quad \varepsilon = \frac{1}{2}, \quad \delta = \frac{1}{2}$$

shows that they do not yield any integrable case.

Exactly as in the cases of the free end lattice, we have been able to calculate the constants of motion for the five distinct cases for which the fixed end Toda Lattice was found to possess the Painlevé property. At this point, we can remark that the leading power of the velocities of each constant of motion can be predicted from group theoretical considerations (see section III). It goes without saying that such a knowledge is most helpful for the direct calculation of the integrals of motion.

IV. - INTEGRABILITY OF THE TODA CHAIN BY LIE ALGEBRA TECHNIQUES

a. - The free end case

We consider the Hamiltonian system in the 6 - dimensional phase space p_i, q_i :

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,,\quad \dot{q}_i = \frac{\partial H}{\partial p_i}\,,\quad H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2} + \exp\left(\varepsilon\,(q_1 - q_2)\right) + \exp\,(q_2$$

After an apropriate canonical transformation we cast it in a form suitable for the search of integrable cases by Lie algebra theory [9]. The cases predicted by the Painlevé analysis and already found by direct calculation of the integrals will appear as the ones corresponding to simple Lie algebras of rank 2 : A_2, B_2, G_2.

We consider the transformation, as in § II :

$$\begin{aligned} q_1' &= \varepsilon\,(q_1 - q_2) \\ q_2' &= q_2 - q_3 \\ q_3' &= m_1 q_1 + m_2 q_2 + q_3 \end{aligned}$$

which we complete to a canonical one by introducing the momenta p_1', p_2', p_3' satisfying $\{q_i', p_j'\} = \delta ij$ ($\{..\}$ denotes the Poisson bracket relative to q_i, p_i coordinates) i.e. :

$$p_1' = \frac{1}{\varepsilon M}\,(M\,p_1 - m_1\,P),\quad p_2' = \frac{1}{M}\,(P - Mp_3),\quad p_3' = \frac{P}{\sqrt{M}}\,,\quad \text{where}$$

$$P = p_1 + p_2 + p_3,\quad M = m_1 + m_2 + 1.$$

Then the reduced Hamiltonian system in the 4 - dimensional phase space reads :

$$p_i' = -\frac{\partial H'}{\partial q_i'}\,,\quad q_i' = \frac{\partial H'}{\partial p_i'}\,,\quad H' = \frac{1}{2}\Big(\sum_{\substack{i=1,2\\ j=1,2}} a_{ij}\,p_i'\,p_j'\Big) + \exp q_1' + \exp q_2'$$

where $a_{11} = \frac{\varepsilon^2 (M - 1)}{m_1 m_2}$, $a_{12} = a_{21} = - \frac{\varepsilon}{m_2}$, $a_{22} = \frac{1 + m_2}{m_2}$

From now on, we will drop the primes in order to alleviate the notations.

The above Hamiltonian system has a Lax form representation furnishing the integrals and providing complete integrability, once the matrix $A = (a_{ij})$ is a positive scalar multiple of the matrix $((h_i, h_j))$, where h_1, h_2 constitute a root system base of a simple Lie algebra of rank 2 and $(.,.)$ denotes the scalar product determined by the Killing form. In other words, the matrix $C = (c_{ij} = \frac{2a_{ij}}{a_{jj}})$ must be a rank 2 Cartan matrix [9].

So, one has to test the values of m_1, m_2 for C to become a rank 2 Cartan matrix i.e. one of the following matrices or their transpose [14] :

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} .$$

In each case ε disappears with time scaling. The cases corresponding to transpose matrices are equivalent within scaling of ε and interchanging p_1 with p_2 and q_1 with q_2. In fact, three distinct integrable cases appear :

Lie Algebra	Cartan matrix	m_1, m_2	Hamiltonian (ε - free)	Squared time scaling factor
A_2	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$	$m_1 = \frac{\varepsilon (2\varepsilon - 1)}{2 - \varepsilon}$ $m_2 = 2\varepsilon - 1$	$2p_1^2 - 2p_1 p_2 + 2p_2^2 + \exp q_1 + \exp q_2$	$\frac{\varepsilon}{2 (2\varepsilon - 1)}$
B_2	$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$	$m_1 = \frac{\varepsilon (2\varepsilon - 1)}{1 - \varepsilon}$ $m_2 = 2\varepsilon - 1$	$p_1^2 - 2p_1 p_2 + 2p_2^2 + \exp q_1 + \exp q_2$	$\frac{\varepsilon}{2 (2\varepsilon - 1)}$
G_2	$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$	$m_1 = \frac{3\varepsilon (2\varepsilon - 1)}{2 - 3\varepsilon}$ $m_2 = 2\varepsilon - 1$	$2p_1^2 - 6p_1 p_2 + 6p_2^2 + \exp q_1 + \exp q_2$	$\frac{\varepsilon}{6 (2\varepsilon - 1)}$

The Lax pair form representation is obtained through an exact representation of minimal dimension of the corresponding Lie algebra i.e. the Hamiltonian system is presented in the form $L = [L.P]$, where $[L, P] = LP - PL$ L,P belong to the representation, and are functions of p_i, q_i.

It follows that the Hamiltonian system has integrals : $I_k = Tr\ (L^k)$, $k = 1, 2, \ldots$.

However we have to perform a subtle search in order to find the algebraically independent ones. In fact, the theory of polynomial invariants [15] gives the values of k : In the case of the Lie algebra A_2 the algebraically independent integrals are I_2 and I_3, for B_2, I_2 and I_4 and for G_2, I_2 and I_6. In every case I_2 is the Hamiltonian (up to a multiplicative constant).

For each algebra A_2, B_2, G_2, a suitable representation can be obtained through their correspondance to $sl(3,C)$, $so(5)$ and $so(7)$ [14].

The only problem that remains is to exhibit the specific form of the Lax Pair (L,P) in these different cases. We will follow for this the method exposed in [9].

Case A_2. (The classical Toda chain).

After changing the variables q_1, q_2 to

$$l_1 = \exp \frac{1}{2} q_1$$

$$l_2 = \exp \frac{1}{2} q_2 ,$$

the equations of motion read :

$$\dot{l_1} = l_1 (2p_1 - p_2), \quad \dot{p_1} = - l_1^2$$

$$\dot{l_2} = l_2 (2p_2 - p_1) \quad \dot{p_2} = - l_2^2$$

which is a suitable form for the search of the Lax Pair. Following [9] we know that the vectors :

$$l(t) = l_1(t) (e_{\alpha_1} + e_{-\alpha_1}) + l_2(t) (e_{\alpha_2} + e_{-\alpha_2}) + p_1 h_1 + p_2 h_2$$

$$A(l(t)) = l_1(t) (e_{\alpha_1} - e_{-\alpha_1}) + l_2(t) (e_{\alpha_2} - e_{-\alpha_2})$$

satisfy the following relation, in the algebra A_2 : $\dot{l} = [l, A(l)]$ where the vectors (e_{α_i}, h_k) are chosen from the basis of the algebra A_2. If we now use the usual matrix representation of A_2, namely $sl(3,C)$ we find the form of the Lax Pair :

$$L = \begin{bmatrix} p_1 & \frac{1}{2} l_1 & 0 \\ l_1 & p_2 - p_1 & \frac{1}{2} l_2 \\ 0 & l_2 & - p_2 \end{bmatrix} , \qquad P = \begin{bmatrix} 0 & \frac{1}{2} l_1 & 0 \\ - l_1 & 0 & \frac{1}{2} l_2 \\ 0 & - l_2 & 0 \end{bmatrix}$$

The algebraically independent integrals are : $I_2 = Tr\ (L^2) = H$

$$\text{et } I_3 = Tr\ (L^3) = 3p_1 p_2 (p_1 - p_2) + \frac{3}{2} (p_2 l_1^2 - p_1 l_2^2)$$

Case B_2. With the same transformation, as in case A_2, the equations of motion are :

$$\dot{l}_1 = l_1 (p_1 - p_2), \quad \dot{p}_1 = - l_1^2$$

and

$$\dot{l}_2 = l_2 (2p_2 - p_1), \quad \dot{p}_2 = - l_2^2$$

Using a 5 x 5 matrix representation of B_2 we write the Lax pair :

$$L = \begin{bmatrix} p_2 & \frac{1}{2} l_2 & 0 & 0 & 0 \\ l_2 & p_1 - p_2 & \frac{1}{2} l_1 & 0 & 0 \\ 0 & l_1 & 0 & \frac{1}{2} l_1 & 0 \\ 0 & 0 & - l_1 & p_2 - p_1 & - \frac{1}{2} l_2 \\ 0 & 0 & 0 & - l_2 & - p_2 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & \frac{1}{2} l_2 & 0 & 0 & 0 \\ - l_2 & 0 & \frac{1}{2} l_1 & 0 & 0 \\ 0 & - l_1 & 0 & - \frac{1}{2} l_1 & 0 \\ 0 & 0 & l_1 & 0 & - \frac{1}{2} l_2 \\ 0 & 0 & 0 & l_2 & 0 \end{bmatrix}$$

The algebraically independent integrals are :

$$I_2 = Tr (L^2) = 2H \quad \text{and} \quad I_4 = Tr (L^4) = 2 (p_1 - p_2)^4 + 2p_2^4 + 2l_1^2 (p_1 - p_2)^2 + 2p_2^2 l_2^2 + 2p_1^2 l_2^2 + 2l_1^2 l_2^2 + l_2^4$$

Note that $I_3 = Tr (L^3) = 0$.

Case G_2. The Hamiltonian system after the transformation $l_1 = \exp \frac{q_1}{2}$, $l_2 = \exp \frac{q_2}{2}$ reads :

$$\dot{l}_1 = l_1 (2p_1 - 3p_2), \quad \dot{p}_1 = - l_1^2$$

$$\dot{l}_2 = l_2 (6p_2 - 3p_1), \quad \dot{p}_2 = - l_2^2$$

and using the 7 x 7 matrix representation of the exceptional Lie algebra G_2 we obtain the Lax pair :

$$L = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}l_1 & 0 & 0 & -l_1\sqrt{2} & 0 & 0 \\ \sqrt{2}\,l_1 & 3p_1-2p_2 & 0 & 3l_2 & 0 & 0 & 0 \\ 0 & 0 & p_1 & 0 & 0 & 0 & -\frac{l_1}{2} \\ 0 & \frac{l_2}{2} & 0 & p_1-3p_2 & 0 & \frac{l_1}{2} & 0 \\ -\frac{\sqrt{2}}{2}l_1 & 0 & 0 & 0 & 2p_1-3p_2 & 0 & -\frac{l_2}{2} \\ 0 & 0 & 0 & l_1 & 0 & -p_1 & 0 \\ 0 & 0 & -l_1 & 0 & -3l_2 & 0 & 3p_2-p_1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}l_1 & 0 & 0 & 0 & \sqrt{2}\,l_1 & 0 \\ -\sqrt{2}\,l_1 & 0 & 0 & 0 & 3l_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{l_1}{2} \\ 0 & -\frac{l_2}{2} & 0 & 0 & 0 & \frac{l_1}{2} & 0 \\ -\frac{\sqrt{2}}{2}l_1 & 0 & 0 & 0 & 0 & 0 & -\frac{l_2}{2} \\ 0 & 0 & 0 & -l_1 & 0 & 0 & 0 \\ 0 & 0 & l_1 & 0 & -3l_2 & 0 & 0 \end{bmatrix}$$

The algebraically independent integrals are $I_2 = Tr\,(L^2) = 6H$ and $I_6 = Tr(L^6)$.

b. The fixed-end case

We consider the Hamiltonian system in the 4-dimensional phase space p_i, q_i :

$$\dot{p}_i = -\frac{\partial H}{\partial q_1}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad H = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2}\right) + \exp(-\delta q_1) + \exp \varepsilon(q_1 - q_2) + \exp q_2$$

After the canonical transformation $q_i' = q_i \sqrt{m_i}$, $q_i' = p_i / \sqrt{m_i}$ and dropping the primes for convenience of notation, the system is written :

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \exp\left(-\frac{\delta}{\sqrt{m_1}} q_1\right) + \exp \varepsilon\left(\frac{q_1}{\sqrt{m_1}} - \frac{q_2}{\sqrt{m_2}}\right) + \exp\frac{q_2}{\sqrt{m_2}}$$

Considering more recent works [12], in order to find completely integrable cases via a Lax-pair form, we have to keep in mind an extension of Bogoyavlenski's theorem presented in [9], within the framework of Kač-Moody algebras.

We will give a brief description of the concept of these algebras based on ref [16] so as to provide the necessary tools in order to obtain the Lax-pair forms.

So, consider a complex simple Lie algebra g and an automorphism σ of g of finite order d [i.e. the least positive integer d such that σ^d = identity] induced by a symmetry of the root system of g. The order d can take the following values :

d = 1 σ = identity , case of all simple Lie algebras
d = 2 case of A_n ; $n \geq 2$, D_n ; $n > 4$, E_6
d = 3 case of D_4

Then one has the following decomposition of g into a direct sum of subspaces : $g = g_0 + g_1 + \ldots + g_{d-1}$ indexed over the integers modulo d [g_k is the eigenspace of σ corresponding to the eigenvalue ε^k ; $\varepsilon = \exp(2\pi i/d)$] with the property : $[g_i, g_j] \subseteq g_{i+j}$; i,j,i+j are integers modulo d.

Especially, g_o is a simple complex Lie algebra namely :

g	g	$A_{2n}(n\geq 1)$	$A_{2n-1}(n\geq 2)$	$D_{n+1}(n>3)$	E_6	D_4
d	1	2	2	2	2	3
g_o	g	B_n	C_n	B_n	F_4	G_2

and because of the relation $[g_o, g_i] \subseteq g_i$, we have a representation of g_o on each g_i which is irreducible. Then, the generalisation of Bogoyavlenski's result consists in considering $-\theta$, the opposite of the highest weight θ of the representation of G_o on G_1 relative to a basis $\alpha_1,\ldots,\alpha_n$ of roots of G_o with respect to a Cartan subalgebra $\underline{h}_o$ of g_o [n is the rank of g_o] and the analogue of Bogoyavlenski's important "admissible" set of roots is the set $\alpha_1,\ldots,\alpha_n$, $\alpha_{n+1} = -\theta$ (θ being the highest weight of the representation of g_o on g_1, when l_i are non-negative integers with at least one of them non-vanishing, $\theta_1 + l_1\alpha_1 + l_2\alpha_2 + \ldots + l_n\alpha_n$ is no more a weight of this representation). This is indeed a generalisation, since, in case d = 1, $\tilde{g} = g_o = g_1$, the representation is the adjoint one and the highest weight is the highest root.

Consider now, as in [9], e_{α_i}, $e_{-\alpha_i}$; i = 1,...,n vectors in g_o, $e_{-\theta}$ in g_1, e_θ in g_1, [there is a duality between the representations of g_o on g_1 and g_{-1}] and a basis $h_1,\ldots,h_n$ of $\underline{h}_o$, that satisfy, among other relations :

$$[e_\lambda, e_{-\lambda}] = \lambda \quad ,\lambda = \alpha_1,\ldots,\alpha_{n+1}$$
$$[h_k, e_\lambda] = (h_k,\lambda)\, e_\lambda, [h_i, h_j] = 0 \text{ (where the scalar product}$$
$$(x,y) = \mathrm{Tr}(\mathrm{adx}\ \mathrm{ady}),\ \mathrm{adx}(z) = [x,z] \ ,\ z \in g)$$

and $[e_{\alpha_i}, e_{-\alpha_j}] = 0$, $i \neq j$ $i,j = 1,2,\ldots,n+1$

Then the theorem 1, in [9] holds i.e. :

If $\alpha_i = \sum_{k=1}^{n} d_{ik} h_k$, i = 1,...,n+1, the Hamiltonian system :

$$\dot{p}_i = -\partial H/\partial q_i \ ,\ \dot{q}_i = \partial H/\partial p_i$$

$$H = \sum_{k,m}^{n} (h_k,h_m)\, p_k p_m + \sum_{j}^{n+1} \exp\left(\sum_{k}^{n} a_{jk}\, q_k\right)$$

has a Lax-pair form.

We present the equation of the Lax-pair into a different form due to Manakov [17] :

$$\dot{l} = [l, A(l)]$$

$$l = \sum_k^n P_k h_k + \sum_j^{n+1} l_j e_{\alpha_j} + \sum_j^{n+1} e_{-\alpha_j}, \quad A(l) = \sum_j^{n+1} l_j e_{\alpha_j}, \quad l_j = \exp\left(\sum_k^n d_{jk} q_k\right)$$

and any linear representation T of g determines the matrix pair $T(l)$, $T(A(l))$.

Applying this theorem in the fixed-end case of the Toda lattice with two movable particles, we obtain completely integrable cases for those values of the parameters m_1, m_2, δ, ε for which the vectors :

$$\beta_1 = -\frac{\delta}{\sqrt{m_1}} h_1, \quad \beta_2 = \frac{1}{\sqrt{m_2}} h_2, \quad \beta_3 = \varepsilon \left(\frac{h_1}{\sqrt{m_1}} - \frac{h_2}{\sqrt{m_2}}\right)$$

form the "admissible" set of roots and weights of a rank 2 simple Lie algebra g_0 which is the σ-invariant subalgebra of a simple Lie algebra g with respect to an automorphism σ of g [h_1, h_2, constitute an orthonormal basis of the Euclidean 2-dimensional space].

Instead of using directly generalised Cartan matrices or extended Dynkin diagrams, we will find the values of the parameters m_1, m_2, δ, ε that provide integrable cases by visualising the "admissible" sets of roots and weights. To begin with, we exclude the $A_2^{(1)}$ system because it contains no pairs of orthogonal vectors as required by the orthogonality of β_1, β_2. We consider the systems $B_2^{(1)}$, $G_2^{(1)}$, $A_4^{(2)}$, $D_4^{(3)}$ (notation of [16]). (Figure 1).

We, then, find the following cases :

Conditions on $\beta_1, \beta_2, \beta_3$	Values of the parameters $\varepsilon, \delta, m_1/m_2$	Hamiltonian
1. $\lvert\beta_1\rvert = \lvert\beta_2\rvert$, $\beta_1 + \beta_2 + 2\beta_3 = 0$	1/2, 1, 1	$\frac{1}{2}(p_1^2 + p_2^2) + e^{-q_1} + e^{\frac{q_1 - q_2}{2}} + e^{q_2}$
2. $3\lvert\beta_1\rvert^2 = \lvert\beta_2\rvert^2$, $3\beta_1 + \beta_2 + \beta_3 = 0$	1/2, 1/3, 1/3	$\frac{1}{2}(p_1^2 + p_2^2) + e^{\frac{-q_1}{3}} + e^{\frac{q_1 - q_2}{2}} + e^{q_2}$

Conditions on β_1,β_2,β_3 Values of the parameters ε,δ, m_1/m_2	Hamiltonian
2 bis. $3\lvert\beta_2\rvert^2 = \lvert\beta_1\rvert^2$, $\beta_1 + 3\beta_2 + \beta_3 = 0$ 3/2, 3, 3	$\frac{1}{2}(\frac{p_1^2}{3} + p_2^2) + e^{-3q_1} + e^{\frac{3(q_1-q_2)}{2}} + e^{q_2}$
3. $2\lvert\beta_1\rvert^2 = \lvert\beta_2\rvert^2$, $2\beta_1 + \beta_2 + \beta_3 = 0$ 1/2, 1/2, 1	$\frac{1}{2}(p_1^2 + p_2^2) + e^{\frac{-q_1}{2}} + e^{\frac{q_1-q_2}{2}} + e^{q_2}$
3 bis. $2\lvert\beta_2\rvert^2 = \lvert\beta_1\rvert^2$, $\beta_1 + 2\beta_2 + \beta_3 = 0$ 1, 2, 1	$\frac{1}{2}(p_1^2 + p_2^2) + e^{-2q_1} + e^{q_1-q_2} + e^{q_2}$
4. $\lvert\beta_1\rvert = \lvert\beta_2\rvert$, $\beta_1 + \beta_2 + \beta_3 = 0$ 1, 1, 1	$\frac{1}{2}(p_1^2 + p_2^2) + e^{-q_1} + e^{q_1-q_2} + e^{q_2}$
5. $3\lvert\beta_1\rvert^2 = \lvert\beta_2\rvert^2$, $\beta_1 + \beta_2 + 2\beta_3 = 0$ 1/2, 1, 3	$\frac{1}{2}(\frac{p_1^2}{3} + p_2^2) + e^{-q_1} + e^{\frac{q_1-q_2}{2}} + e^{q_2}$
5 bis. $3\lvert\beta_2\rvert^2 = \lvert\beta_1\rvert^2$, $\beta_1 + \beta_2 + 2\beta_3 = 0$ 1/2, 1, 1/3	$\frac{1}{2}(p_1^2 + \frac{p_2^2}{2}) + e^{-q_1} + e^{\frac{q_1-q_2}{2}} + e^{q_2}$

Case 2 is equivalent to 2 bis by the transformation $q_1 \to -3q_2$, $q_2 \to -3q_1$, plus a scaling in time. Case 3 is equivalent to 3 bis by : $q_1 \to -q_2/2$, $q_2 \to -q_1/2$ plus a scaling in time and 5 to 5 bis by $q_1 \to -q_2$, $q_2 \to -q_1$.

Let now give the Lax-pair form representations for each of the non-equivalent above cases and the indication of the algebraically independent integrals.

Case $B_2^{(1)}$. We consider the Hamiltonian

$$H = \frac{1}{2}(P_1^2 + p_2^2) + \exp(-2q_1) + \exp(q_1 - q_2) + \exp(2q_2)$$

(after a scaling).
The equations of motion after the transformation :
$l_1 = \exp(-2q_1)$, $l_2 = \exp 2q_2$, $l_3 = \exp(q_1 - q_2)$, are written

$$\dot{l}_1 = -2\, l_1 p_1 \qquad \dot{p}_1 = 2l_1 - l_3$$

$$\dot{l}_2 = 2\, l_2 p_2 \qquad \dot{p}_2 = l_3 - 2l_2$$

$$\dot{l}_3 = l_3 (p_1 - p_2)$$

Using the standard representation $so(5)$ of B_2 and, identifying the matrices corresponding to a basis of the roots and the opposite of the highest root, we obtain the Lax-pair :

$$L = \begin{array}{|c|c|c|c|c|} \hline -(p_1+p_2) & l_1 & & 2 & \\ \hline 2 & p_1-p_2 & l_3 & & -2 \\ \hline & 2 & & -l_3 & \\ \hline l_2 & & -2 & p_2-p_1 & -l_1 \\ \hline & -l_2 & & -2 & p_1+p_2 \\ \hline \end{array}, \quad P = \begin{array}{|c|c|c|c|c|} \hline & l_1 & & & \\ \hline & & l_3 & & \\ \hline & & & -l_3 & \\ \hline l_2 & & & & -l_1 \\ \hline & -l_2 & & & \\ \hline \end{array}$$

The algebraically independent integrals are $I_2 = Tr(L^2) = 8\,H$ and $I_4 = Tr(L^4)$

Case $G_2^{(1)}$. The Hamiltonian is :

$$H = \frac{1}{4}\left(\frac{p_1^2}{3} + p_2^2\right) + \exp(-6q_1) + \exp 3(q_1-q_2) + \exp 2q_2$$

(after a scaling) and with the transformation :

$$l_1 = \exp(-6q_1)$$
$$l_2 = \exp(2q_2)$$
$$l_3 = \exp 3(q_1 - q_2)$$

The equations of motion are :

$$\dot{l}_1 = -\,l_1 p_1 \qquad \dot{p}_1 = 6l_1 - 3l_3$$

$$\dot{l}_2 = l_2 p_2 \qquad \dot{p}_2 = 3l_3 - 2l_2$$

$$\dot{l}_3 = \frac{1}{2}\,l_3\,(p_1 - 3p_2)$$

and the Lax pair, provided by the 7 x 7 matrix representation of G_2 is :

$L =$

$\frac{p_1-p_2}{2}$	l_3	3		1		
3	p_2		$l_2\sqrt{2}$			
l_1		$-\frac{p_1+p_2}{2}$				-1
	$\sqrt{2}$				$-l_2\sqrt{2}$	
l_2				$\frac{p_1+p_2}{2}$		-3
			$-\sqrt{2}$		$-p_2$	$-\,l_3$
		$-\,l_2$		$-\,l_1$		$\frac{p_2-p_1}{2}$

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline & l_3 & & & & & \\ \hline & & & l_2\sqrt{2} & & & \\ \hline l_1 & & & & & & \\ \hline & & & & & -l_2\sqrt{2} & \\ \hline l_2 & & & & & & \\ \hline & & & & & & -l_3 \\ \hline & & -l_2 & & -l_1 & & \\ \hline \end{array}$$

The algebraically independent integrals are $I_2 = \mathrm{Tr}(L^2) = \frac{3}{2} H$ and $I_6 = \mathrm{Tr}(L^6)$.

Case $A_4^{(2)}$. The Hamiltonian is

$$H = \frac{1}{2} (p_1^2 + p_2^2) + \exp(-2q_1) + \exp(q_1 - q_2) + \exp q_2$$

with the transformation : $l_1 = \exp(-2q_1)$, $l_2 = \exp q_2$, $l_3 = \exp(q_1 - q_2)$ the equations of motion write :

$$\dot{l}_1 = 2l_1p_1 \qquad \dot{p}_1 = 2l_1 - l_3$$

$$\dot{l}_2 = l_2p_2 \qquad \dot{p}_2 = l_3 - l_2$$

$$\dot{l}_3 = l_3 (p_1 - p_2)$$

We will use the standard 5 x 5 matrix representation sl(5) of A_4. An automorphism of order two (involution) induced, by a symmetry of the root system of A_4 is given by :

$$\begin{array}{|c|c|c|c|c|} \hline a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ \hline \end{array} \quad \xrightarrow{\sigma} \quad \begin{array}{|c|c|c|c|c|} \hline -a_{55} & a_{45} & -a_{35} & a_{25} & -a_{15} \\ \hline a_{54} & -a_{44} & a_{34} & -a_{24} & a_{14} \\ \hline -a_{33} & a_{43} & -a_{33} & a_{23} & -a_{13} \\ \hline a_{52} & -a_{42} & a_{32} & -a_{22} & a_{12} \\ \hline -a_{51} & a_{41} & -a_{31} & a_{21} & -a_{11} \\ \hline \end{array}$$

and the matrices M belonging to the B_2 [σ-invariant] subalgebra are easily identified as the ones of the form :

$M =$

b_{11}	b_{12}	b_{13}	b_{14}	
b_{21}	b_{22}	b_{23}		b_{14}
b_{31}	b_{32}		b_{23}	$-b_{13}$
b_{41}		b_{32}	$-b_{22}$	b_{12}
	b_{41}	$-b_{31}$	b_{21}	$-b_{11}$

and the one dimensional subspace of the matrices corresponding to the opposite of the highest weight of the representation of g_o on g_1 are the matrices (a_{ij}) $1 \leq i,j \leq 6$ with all entries 0 except a_{51}.

So a Lax-pair for the considered Hamiltonian system is :

$L =$

p_1	l_3			2
1	p_2	l_2		
	1		l_2	
		1	$-p_2$	l_3
l_1			1	$-p_1$

, $P =$

	l_3			
		l_2		
			l_2	
				l_3
l_1				

The algebraically independent integrals are : $I_2 = Tr\,(L^2) = 4\,H$ and $I_4 = Tr(L^4)$

Case $D_3^{(2)}$. The Hamiltonian is :

$$H = p_1^2 + p_2^2 + \exp(-q_1) + \exp(q_1 - q_2) + \exp q_2$$

with $l_1 = \exp(-q_1)$, $l_2 = \exp q_2$, $l_3 = \exp(q_1 - q_2)$

The equations of motion write :

$$\dot{l}_1 = -2l_1p_1 \qquad \dot{p}_1 = l_1 - l_3$$

$$\dot{l}_2 = 2\,l_2p_2 \qquad \dot{p}_2 = l_3 - l_2$$

$$\dot{l}_3 = 2\,l_3\,(p_1 - p_2)$$

We use the 4 x 4 standard representation sl(4) of the $D_3 \simeq A_3$ with the automorphism of order two :

a_{11}	a_{12}	a_{13}	a_{14}		$-a_{44}$	a_{34}	$-a_{24}$	a_{14}
a_{21}	a_{22}	a_{23}	a_{24}	$\overset{\sigma}{\rightarrow}$	a_{43}	$-a_{33}$	a_{23}	$-a_{13}$
a_{31}	a_{32}	a_{33}	a_{34}		$-a_{42}$	a_{32}	$-a_{22}$	a_{12}
a_{41}	a_{42}	a_{43}	a_{44}		a_{41}	$-a_{31}$	a_{21}	$-a_{11}$

to identify the matrices-elements of the σ-invariant subalgebra of type B_2. These matrices have the following form :

b_{11}	b_{12}	b_{13}	b_{14}
b_{21}	b_{22}	b_{23}	$-b_{13}$
b_{31}	b_{32}	$-b_{22}$	b_{12}
b_{41}	$-b_{31}$	b_{23}	$-b_{11}$

and the one dimensional space of matrices corresponding to the opposite of the highest weight are the matrices (a_{ij}) ; $1 \leq i,j \leq 4$ with the entries vanishing except $a_{31} = \alpha_{42}$.

The Lax-pair for the Hamiltonian system is :

$$L = \begin{array}{|c|c|c|c|} \hline -p_1-p_2 & l_1 & 1 & \\ \hline 1 & p_1-p_2 & l_3 & 1 \\ \hline l_2 & 2 & p_2-p_1 & l_1 \\ \hline & l_2 & 1 & p_1+p_2 \\ \hline \end{array} \quad , \quad P = \begin{array}{|c|c|c|c|} \hline & l_1 & & \\ \hline & & l_3 & \\ \hline l_2 & & & l_1 \\ \hline & l_2 & & \\ \hline \end{array}$$

The algebraically independent integrals are : $I_2 = \mathrm{Tr}\,(L^2) = 4H$ and $I_4 = \mathrm{Tr}\,(L^4)$.

Case $D_4^{(3)}$. The Hamiltonian is

$$H = \frac{1}{4}\,(3p_1^2 + p_2^2) + \exp\,(-2q_1) + \exp\,(q_1 - q_2) + \exp\,2\,q_2$$

and the equations of motion write :

$$\dot{l}_1 = -\,3\,l_1 p_1 \qquad \dot{p}_1 = 2\,l_1 - l_3$$

$$\dot{l}_2 = l_2 p_2 \qquad \dot{p}_2 = l_3 - 2\,l_2$$

$$\dot{l}_3 = \frac{1}{2}\,l_3\,(3\,p_1 - p_2)$$

We consider the 8 x 8 standard representation $\mathbf{so}(8)$ of D_4 with the automorphism σ of order three :

a_{11}	a_{12}	a_{13}	a_{14}	$-b_{11}$	$-b_{12}$	$-b_{13}$	
a_{21}	a_{22}	a_{23}	a_{24}	$-b_{21}$	$-b_{22}$		b_{13}
a_{31}	a_{32}	a_{33}	a_{34}	$-b_{31}$		b_{22}	b_{12}
a_{41}	a_{42}	a_{43}	a_{44}		b_{31}	b_{21}	b_{11}
$-c_{11}$	$-c_{12}$	$-c_{13}$		$-a_{44}$	$-a_{34}$	$-a_{24}$	$-a_{14}$
$-c_{21}$	$-c_{22}$		c_{13}	$-a_{43}$	$-a_{33}$	$-a_{23}$	$-a_{13}$
$-c_{31}$		c_{22}	c_{12}	$-a_{42}$	$-a_{32}$	$-a_{22}$	$-a_{12}$
	c_{31}	c_{21}	c_{11}	$-a_{41}$	$-a_{31}$	$-a_{21}$	$-a_{11}$

$\downarrow \sigma$

a'_{11}	b_{31}	$-b_{21}$	b_{11}	$-b_{22}$	$-b_{12}$	$-b_{13}$	
c_{13}	a_{22}'	a_{23}	$-a_{13}$	$-a_{24}$	$-a_{14}$		b_{13}
$-c_{12}$	a_{32}	a_{33}'	a_{12}	$-a_{34}$		a_{14}	b_{12}
c_{11}	$-a_{31}$	a_{21}	a'_{44}		a_{34}	a_{24}	b_{22}
$-c_{22}$	$-a_{42}$	$-a_{43}$		$-a'_{44}$	$-a_{12}$	a_{13}	$-b_{11}$
$-c_{21}$	$-a_{41}$		a_{43}	$-a_{21}$	$-a'_{33}$	$-a_{23}$	b_{21}
$-c_{31}$		a_{41}	a_{12}	a_{31}	$-a_{32}$	$-a'_{22}$	$-b_{31}$
	c_{31}	c_{21}	c_{22}	$-c_{11}$	b_{12}	$-b_{13}$	$-a'_{11}$

where : $a'_{11} = \frac{1}{2}(a_{11} + a_{22} + a_{33} - a_{44})$, $a'_{22} = \frac{1}{2}(a_{11} + a_{22} - a_{33} + a_{44})$,

$a'_{33} = \frac{1}{2}(a_{11} - a_{22} + a_{33} + a_{44})$, $a'_{44} = -\frac{1}{2}(-a_{11} + a_{22} + a_{33} + a_{44})$

Following [16] we exhibit the matrices corresponding to the four basic roots of D_4 i.e.

matrix	non vanishing elements of the matrix
X_1	a_{12}
X_2	a_{23}
X_3	a_{34}
X_4	b_{31}

Then, the matrices corresponding to basic roots in the σ-invariant subalgebra of type G_2 are : X_2 and $X_1 + X_3 + X_4$, and the one-dimensional subspace corresponding to the highest weight is spanned by : $[X_1,[X_2,X_3]] + \varepsilon\,[X_3,[X_2,X_4]] + \varepsilon^2\,[X_4,[X_2,X_1]]$ (so the space corresponding to the opposite of the highest weight is spanned by the transpose of this matrix) where $\varepsilon = \exp(2\,i/3)$. Finally the Lax-pair for the Hamiltonian system is :

$L =$

$-p_2$	l_2		1	$-\varepsilon$			
1	$\frac{-p_2-3p_1}{2}$	l_1			$-\varepsilon^2$		
	3	$\frac{3p_1-p_2}{2}$	l_2	$-l_2$	ε^2		
l_3		1			l_2		ε
$-\varepsilon^2 l_3$		-1		$-l_2$			-1
	$-\varepsilon l_3$		1	-1	$\frac{p_2-3p_1}{2}$	$-l_1$	
		εl_3				$\frac{p_2+3p_1}{2}$	$-l_2$
			$\varepsilon^2 l_2$	$-l_3$		-1	p_2

,

$$P = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & I_2 & & & & & & \\
\hline
 & & I_1 & & & & & \\
\hline
 & & & I_2 & -I_2 & & & \\
\hline
I_3 & & & & & I_2 & & \\
\hline
-\varepsilon^2 I_3 & & & & & -I_2 & & \\
\hline
 & -\varepsilon I_3 & & & & & -I_1 & \\
\hline
 & & -\varepsilon I_3 & & & & & -I_2 \\
\hline
 & & & \varepsilon^2 I_3 & -I_3 & & & \\
\hline
\end{array}$$

The algebraically independent integrals are

$$I_2 = \mathrm{Tr}\,(L^2) = 12\,H \quad \text{and} \quad I_6 = \mathrm{Tr}\,(L^6).$$

CONCLUSION :

In this work, we have presented some new integrable cases for systems which are restrictions of the Toda lattice to chains of two or three particles, interacting exponentialy with their nearest neighbors. This restriction of dimensionality leads to new integrable cases as it allows the choice of unequal masses and unequal ranges of interaction. The Painlevé analysis of Bountis et al and of A. Ramani has identified the different values of parameters for which the above systems could possess a second integral of motion apart from the energy. The integrability of all these cases has been demonstrated in this work through two distinct approaches : Direct computation of the integrals of motion and group theoretical methods. The first method , whenever practicable, allows at the same time, the identification of the integrable cases and the exact calculation of the integrals of motion. The Lie algebra approach can be used independently from the direct search. It allows easily the identification of integrable cases and can, in principle, be used for the computation of the constant of motion although the latter calculation becomes sometimes pretty involved.

What emerges from our work is that the use of different, independent approaches, such as Painlevé Analysis, Direct search for the constant of motion and group theoretical methods, can be a most powerful tool for the investigation of integrability of dynamical systems.

Aknowledgements

The authors are most grateful to Alfred Ramani for numerous fruitful discussions.

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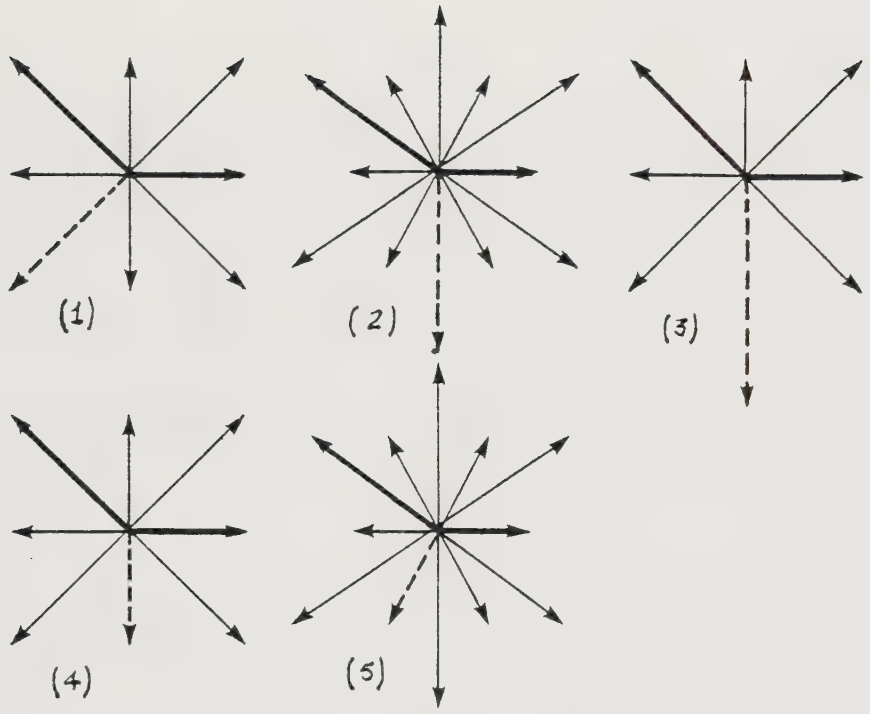

Figure caption

Identification of a basis of roots (heavy lined arrows) and the opposite of the nignest weight (dashed arrow) of the g_0 subalgebra of the simple Lie algebra :

(1) B_2, (2) G_2, (3) A_4, (4) D_3, (5) D_4.

CHAPITRE 4

ANALYSE DES SINGULARITÉS ET INTÉGRABILITÉ : LE CRITÈRE DE PAINLEVÉ

La plupart du temps, lorsque l'on est en présence d'une équation différentielle ordinaire, il est impossible d'écrire la solution sous forme fermée. Cependant, pourvu que les coefficients de l'équation soient suffisamment réguliers, on peut toujours intégrer localement l'équation au voisinage d'une condition initiale donnée. L'équation pourra être considérée comme intégrée si on peut étendre cette solution locale en une solution globale. Il est raisonnable de penser que c'est de la nature des singularités de l'équation (dans le plan complexe du temps) que va dépendre l'existence ou pas d'une telle extension.

Pour les équations différentielles linéaires, la classification des singularités est facilitée par le fait que les singularités des solutions ne viennent que de celles des coefficients de l'équation elle-même (singularités fixes).

Par contre, cette détection est plus délicate dans le cas non linéaire. Il existe d'autres types de singularités, appelées mobiles, qui dépendent des conditions initiales.

Il suffit pour s'en convaincre de considérer l'équation :

$$y' = y^2$$

Sa solution s'écrit $y = \frac{1}{z - z_0}$ et présente un pôle mobile en z_0.

Dans le cas des équations du premier ordre de la forme :

$$\frac{dw}{dz} = \frac{g(z,w)}{h(z,w)}$$

où g et h sont des polynômes en w de coefficients analytiques en z, Painlevé a prouvé que les singularités mobiles étaient nécessairement algébriques [1].

Par contre, dès que l'ordre des équations est plus élevé (deux ou plus), des singularités mobiles essentielles peuvent exister.

Considérons en effet l'équation :

$$(a-1)w'^2 - aww'' = 0$$

de solution $w(z) = C_1(z-C_2)^a$.

Elle a une singularité mobile en $z = C_2$ qui est un point critique transcendant dès que a est irrationnel.

De même l'équation :

$$\left[\frac{d}{dz}\left(\frac{w'}{w}\right)\right]^2 + 4\left(\frac{w'}{w}\right)^3 = 0$$

a pour solution :

$$w(z) = C_1 \exp(z-C_2)^{-1}$$

qui a une singularité mobile essentielle à $z = C_2$.

Avec les équations d'ordre 3, la situation est encore plus compliquée. Il peut même y avoir des frontières naturelles mobiles. Ce cas a été matérialisé récemment par J. Weiss *et al.* au cours de leur étude de l'hamiltonien de Hénon-Heiles [2].

Préoccupons-nous à présent de la question de l'intégrabilité des équations différentielles.

Les premières équations qui ont été étudiées sont les équations linéaires. Elles sont considérées comme intégrables étant donné que même si on ne sait pas exprimer leur solution en terme de fonctions connues, on peut à priori, à la vue de l'équation elle-même, avoir une idée du type de comportement de la solution générale du fait qu'elle s'exprime comme superposition de solutions particulières bien choisies. De nombreuses classes d'équations linéaires ont ainsi servi à définir des fonctions spéciales (exponentielles, Bessel, etc.).

Dans le cas non linéaire, les choses sont plus compliquées par le fait que, comme on l'a dit plus haut, des singularités mobiles peuvent apparaître. Néanmoins, une classe importante de fonctions, les fonctions elliptiques qui sont caractérisées par le fait d'être monovaluées, doublement périodiques avec des pôles comme seules singularités, s'expriment comme solutions d'équations différentielles non linéaires (Briot, Bouquet). Plus généralement, on peut espérer raisonnablement que si les singularités critiques (c'est-à-dire les singularités essentielles ou les coupures algébriques) de l'équation sont fixes et non mobiles, comme dans le cas linéaire, l'équation peut être considérée comme intégrable.

Les travaux de Fuchs (1884) et Liouville montrent que les seules équations du premier ordre de la forme *(1)* qui possèdent cette propriété sont les équations linéaires et les équations de Riccati :

$$\frac{dw}{dz} = a(z) + b(z)w + c(z)w^2$$

Ces dernières se ramènent à une équation linéaire du deuxième ordre et peuvent être considérées comme intégrables par linéarisation.

Peu après, Sonia Kowalevskaya (1888) étudia le mouvement d'un corps fixe autour d'un point fixe [3]. Ce mouvement est décrit par un système de six équations couplées du premier ordre :

$$A \frac{d\omega_1}{dt} = (B-C)\omega_2\omega_3 + z_0\gamma_2 - y_0\gamma_3$$

$$B \frac{d\omega_2}{dt} = (C-A)\omega_3\omega_1 + x_0\gamma_3 - z_0\gamma_1$$

$$C \frac{d\omega_3}{dt} = (A-B)\omega_1\omega_2 + y_0\gamma_1 - x_0\gamma_2$$

$$\frac{d\gamma_1}{dt} = \omega_3\gamma_2 - \omega_2\gamma_3$$

$$\frac{d\gamma_2}{dt} = \omega_1\gamma_3 - \omega_3\gamma_1$$

$$\frac{d\gamma_3}{dt} = \omega_2\gamma_1 - \omega_1\gamma_2$$

où A, B, C, x_0, y_0, z_0 sont six paramètres constants et $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \text{cste}$.

Pour un tel problème, la connaissance de trois autres constantes du mouvement permet de ramener l'intégration à des quadratures.

Deux intégrales existent indépendamment des valeurs des paramètres, et c'est donc l'existence d'une troisième intégrale qui est la clef de l'intégrabilité.

Avant 1888, on connaissait deux cas d'intégrabilité :

$x_0 = y_0 = z_0 = 0$ (cas d'Euler)

$A = B$ et $x_0 = y_0 = 0$ (cas de Lagrange)

Dans chacun de ces cas, l'intégration explicite se fait en termes de fonctions elliptiques et par conséquent chacune des fonctions ω_1, ω_2, ω_3, γ_1, γ_2, γ_3 est monovaluée et ne possède comme seules singularités que des pôles.

C'est en cherchant pour quelles valeurs des paramètres cette dernière propriété est vérifiée que S. Kowalevskaya découvrît un nouveau cas d'intégrabilité. A savoir : $A = B = 2C$ et $z_0 = 0$.

C'était la première fois que l'étude des singularités des solutions dans le plan complexe du temps permettait la découverte de systèmes intégrables.

Dans le même esprit, Painlevé et Gambier, au début du siècle, ont classifié les équations du second ordre de la forme [4] :

$$\frac{d^2\omega}{dz^2} = R\left(z, \omega, \frac{d\omega}{dz}\right) \qquad (2)$$

R : fonction rationnelle de ω et $\frac{d\omega}{dz}$, de coefficients analytiques en z, dont les solutions ne possèdent ni coupures algébriques mobiles, ni singularités essentielles.

Ils ont identifié cinquante classes d'équations qui, moyennant des transformations du type :

$$W = \frac{l(z)\omega + m(z)}{p(z)\omega + q(z)} \; ; \; Z = \phi(z)$$

où l, m, p, q et ϕ sont des fonctions analytiques de z, permettent de classifier toutes les équations de la forme *(2)* à points critiques fixes.

La plupart de ces équations canoniques s'intègrent au moyen de fonctions elliptiques, d'équation de Riccati ou d'équations linéaires. Pour d'autres, l'intégration se ramène à l'intégration de six d'entre elles, irréductibles (c'est-à-dire qu'elles ne se réduisent pas à une équation plus simple ni à une combinaison d'équations plus simples). Ces six équations sont fondamentalement différentes et servent à définir de nouvelles transcendantes. Leurs formes canoniques sont les suivantes :

$$I \; : \; \frac{d^2\omega}{dz^2} = 6\omega^2 + z \qquad\qquad II \; : \; \frac{d^2\omega}{dz^2} = 2\omega^3 + z\omega + \alpha$$

$$III \; : \; \frac{d^2\omega}{dz^2} = \frac{1}{\omega}\left(\frac{d\omega}{dz}\right)^2 - \frac{1}{z}\frac{d\omega}{dz} + \frac{1}{z}(\alpha\omega^2+\beta) + \gamma\omega^3 + \frac{\delta}{\omega}$$

$$IV \; : \; \frac{d^2\omega}{dz^2} = \frac{1}{2\omega}\left(\frac{d\omega}{dz}\right)^2 + \frac{3\omega^2}{2} + 4z\omega^2 + 2(z^2-\alpha)\omega + \frac{\beta}{\omega}$$

$$V \; : \; \frac{d^2\omega}{dz^2} = \left\{\frac{1}{2\omega} + \frac{1}{\omega-1}\right\}\left(\frac{d\omega}{dz}\right)^2 - \frac{1}{z}\frac{d\omega}{dz} + \frac{(\omega-1)^2}{z^2}\left(\alpha\omega + \frac{\beta}{\omega}\right) + \frac{\gamma\omega}{z} + \frac{\delta\omega(\omega+1)}{\omega-1}$$

$$VI \; : \; \frac{d^2\omega}{dz^2} = \frac{1}{2}\left\{\frac{1}{\omega} + \frac{1}{\omega-1} + \frac{1}{\omega-z}\right\}\left(\frac{d\omega}{dz}\right)^2 - \left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right\}\frac{d\omega}{dz}$$
$$+ \frac{\omega(\omega-1)(\omega-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{\omega^2} + \frac{\gamma(z-1)}{(\omega-1)^2} + \frac{\delta z(z-1)}{(\omega-z)^2}\right\}$$

A la vue de ces résultats, il apparaît que l'étude des singularités des solutions des équations différentielles ordinaires dans le plan complexe du temps est un outil très puissant pour prédire l'intégrabilité. Nous allons voir que cette intuition s'avère aussi vraie dans le cadre des équations aux dérivées partielles.

Une telle équation est généralement considérée comme intégrable lorsque l'on a pu ramener sa résolution à celle d'un problème linéaire. Plus précisément, regardons plus en détail le cas de l'équation de Korteweg de Vries(KdV), dont l'intégration est à l'origine de la méthode de diffusion inverse (I.S.T. pour "Inverse Scattering Transform", en anglais).

Elle s'écrit :

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (3)$$

Pour l'intégrer, on introduit un problème de diffusion auxiliaire, c'est-à-dire que l'on considère d'une part le problème aux valeurs propres pour l'équation de Schrödinger :

$$-\Psi_{xx} + u\Psi = \lambda\Psi \qquad (4)$$

où u tend vers zéro à l'infini, lorsque $|x|$ tend vers l'infini, et d'autre part, le problème d'évolution :

$$\Psi_t + 4\Psi_{xxx} - 3u_x\Psi - 6u\Psi_x = 0 \qquad (5)$$

Lorsque u est solution de l'équation *(3)*, ces deux équations *(4)*, *(5)* sont compatibles et λ est de plus indépendant du temps. Il est ensuite possible de reconstruire le potentiel u à partir des éléments propres de l'équation de Schrödinger. On utilise pour cela l'équation intégrale de Gelfand, Levitan, Martchenko (G.L.M.). C'est cette dernière inversion qui a donné son nom à la méthode du problème inverse de la diffusion, méthode qui peut s'énoncer ainsi dans un cadre général :

Considérons une P.D.E. écrite en terme d'une fonction inconnue u. On lui associe deux opérateurs L et H dépendant de u, tels que l'évolution :

$$\Psi_t = H\Psi$$

soit compatible avec le problème aux valeurs propres :

$$L\Psi = \lambda\Psi$$

et λ est indépendant de t si et seulement si u satisfait à la P.D.E. considérée.

Puis, on reconstitue la fonction u à partir des éléments propres de l'opérateur L. Cette dernière étape n'est souvent pas triviale (cf. équation de Benjamin-Ono) et revient généralement à la résolution d'une équation intégrale pour u. Mais déjà la simple existence des opérateurs L et H permet d'assurer un grand nombre de propriétés comme l'existence d'une infinité de quantités conservées ou l'existence de transformations de Bäcklund.

Une conjecture récente [5], proposée par Ablowitz, Ramani, Segur, relie l'intégrabilité des P.D.E. et la structure des singularités dans le plan complexe de certaines O.D.E. associées. Elle s'énonce ainsi :

Toute équation différentielle ordinaire, qui est une réduction exacte d'une équation aux dérivées partielles soluble par la méthode I.S.T. possède la propriété de Painlevé, c'est-à-dire que ses seules singularités mobiles dans le plan complexe du temps sont des pôles.

Enoncée ainsi, la conjecture s'est trouvée vérifiée pour un grand nombre de P.D.E. que l'on savait explicitement solubles (par la méthode I.S.T.). Elle permet aussi de déterminer facilement les équations qui ne seront pas intégrables. Par contre, si on cherche à l'utiliser comme détecteur d'intégrabilité, il faut la prolonger de la manière suivante : Si toutes les réductions d'une P.D.E. donnée ont la propriété de Painlevé, alors l'équation considérée est "intégrable".

Arrivé à ce point, une méthode pour détecter si une O.D.E. possède ou pas la propriété de Painlevé paraît nécessaire. Kowalevskaya et Painlevé, lors des études dont nous venons de parler, ont tous les deux utilisé des développements en série formelle autour des singularités pour étudier la nature de celle-ci.

Dans le même esprit, Ablowitz, Ramani, Segur ont proposé un algorithme systématique [6] qui est aussi assez proche de la technique de Fuchs pour l'étude des points critiques des équations linéaires.

Nous allons maintenant présenter les étapes importantes de cet algorithme qui est exposé en détail dans la publication qui vient à la suite.

Il s'applique aux systèmes de n équations différentielles ordinaires du premier ordre et donc aux équations différentielles d'ordre n écrites sous la forme :

$$\frac{d^n x}{dt^n} = F(t,x,\ldots,x^{(n-1)}) \qquad (6)$$

Remarquons tout de suite que cette méthode permet l'étude des singularités algébriques d'une O.D.E. et ne permet pas de détecter les singularités essentielles. Elle fournira donc des conditions nécessaires pour qu'une équation possède la propriété de Painlevé.

On présuppose que, au voisinage de la singularité mobile t_0, x possède un développement de la forme :

$$x = a(t-t_0)^{\alpha} \sum_{j=0}^{\infty} c_j (t-t_0)^j \qquad (0 < |t-t_0| < R) \qquad (7)$$

Si on veut vérifier que le point t_0 est un pôle de l'équation, on doit d'abord montrer que les seuls démarrages possibles de la série *(7)* sont tels que tous les α soient *des entiers* (étape 1). Pour trouver les valeurs de ces coefficients α, on est amené à ne considérer dans l'équation *(6)* que les termes les plus divergents et à écrire la condition sur α pour qu'ils s'équilibrent. Ces termes sont les termes dominants.

La solution générique d'une équation de type *(6)* dépend à priori de n constantes arbitraires. La deuxième étape va alors consister à déterminer les résonances de l'équation, c'est-à-dire les ordres ρ ($\rho \in R$) où les constantes libres peuvent apparaître dans un développement formel au voisinage de t_0. Bien entendu, l'existence de tels indices irrationnels ou rationnels non entiers,positifs ou complexes de partie réelle positive, est incompatible avec un développement de type *(7)* dans lequel tous les exposants sont des entiers. Les résonances négatives (ou de partie réelle négative) sont considérées comme fictives. Les seules résonances positives admissibles sont les entiers.

Remplaçons alors un développement du type *(7)* dans l'équation *(6)*. A priori, la plupart des coefficients sont déterminés, ordre par ordre, en fonction des précédents. Ceci n'est pas vrai pour les indices j qui correspondent aux résonances entiers positifs de l'équation. Pour ces valeurs de j, deux situations opposées peuvent se présenter : soit on peut effectivement choisir C_j arbitraire dans le développement, soit *aucun* choix de constantes C_j n'est compatible avec un développement de type *(7)*. Dans ce dernier cas, c'est en général un développement avec des logarithmes qui convient, de la forme :

$$x(t) = a(t-t_0)^{\alpha} + c_1(t-t_0)^{\alpha+1} + \ldots + c_j(t-t_0)^{\alpha+j}\,(1 + d_{j,1}\log(t-t_0) + \ldots + d_{jk}\,[\log(t-t_0)]^k + \ldots) + \ldots$$

La première situation est celle d'une équation qui possède la propriété de Painlevé, mais il n'en est bien sûr pas de même pour la seconde.

La troisième étape de l'algorithme consiste à remplacer directement le développement en série, tronqué à la valeur de la résonance dans l'équation pour déterminer si un logarithme s'introduit effectivement ou pas dans la série à cet ordre.

Les différentes étapes de la méthode seront détaillées et précisées dans la publication qui suit.

Telle quelle, la conjecture A.R.S. pour les P.D.E. a permis de sélectionner un grand nombre d'équations intégrables par la méthode I.S.T.

Plus récemment, elle a pu être étendue par Weiss, Tabor et Carnevale [7] de manière à se libérer des réductions aux O.D.E.

Leur approche est basée sur la considération des singularités des solutions de laP.D.E. Ces singularités forment une variété différentiable (à la différence du cas des O.D.E. où les singularités étaient supposées isolées).

On dira alors qu'une P.D.E. possède la propriété de Painlevé si les solutions sont monovaluées au voisinage des variétés formées des singularités mobiles, c'est-à-dire que localement la solution a la forme (dans le cas d'une fonction $u(z_1,z_2)$ de deux variables) :

$$u(z_1,z_2) = X^{\alpha}(z_1,z_2) \sum_{n=0}^{\infty} a_n(z_2)\, X^n(z_1,z_2) \qquad (8)$$

$$X(z_1,z_2) = z_1 + \Psi(z_2) \quad (\alpha \text{ est un entier}) \qquad (9)$$

$X(z_1,z_2) = 0$ est l'équation de la variété des singularités.
a_n et X sont des fonctions analytiques au voisinage de $X(z_1,z_2) = 0$.

Cette forme simplifiée *(9)* a été proposée par M. Kruskal ; elle permet d'alléger beaucoup les calculs qui deviennent à peine plus compliqués que dans le cas des O.D.E.

En effet, vérifions par exemple que l'équation de Burger satisfait au critère de Painlevé :

Cette équation s'écrit :

$$u_t + uu_x = \sigma u_{xx} \qquad (10)$$

On suppose que u s'écrit :

$$u = X^{\alpha} \sum_{n=0}^{\infty} a_n(t)\, X^n$$

avec $X = x + \Psi(t)$.

Cherchons d'abord la valeur du coefficient α :

Si on remplace u par a_0X^{α} dans l'équation *(10)*, les termes dominants sont uu_x et σu_{xx} et on obtient, en demandant que ces deux termes s'équilibrent :

$$\alpha = -1, \quad a_0 = -2\sigma$$

Les résonances sont alors obtenues en remplaçant u par $-\frac{2\sigma}{X}(1+a_n X^n)$ dans l'équation réduite :

$$uu_x = \sigma u_{xx}$$

On obtient alors comme condition sur n :

$$(n+1)(n-2) = 0$$

Les résonances sont donc -1 et 2.

Il reste à vérifier qu'à la résonance 2, des logarithmes ne s'introduisent pas.

Pour cela, on remplace u par :

$$-\frac{2\sigma}{X}(1+a_1 X + a_2 X^2)$$

dans l'équation *(10)*.

On est alors conduit à la relation suivante :

$$\Psi'(t) = 2\sigma a_1$$

La condition sur a_2 est satisfaite automatiquement ; ce qui prouve que l'équation de Burgers vérifie le critère de Painlevé.

On retrouve sous cette forme l'intégrabilité d'équations connues (KdV, mKdV, Bousinesq, Kadomtsev-Petviashvili, etc.) et on peut aussi dans certains cas retrouver la paire de Lax associée à l'équation dans la méthode I.S.T. (c'est-à-dire les opérateurs L et H définis précédemment).

Revenons à présent au cas des O.D.E.

Les résultats de Kowalevskaya et Painlevé montrent l'utilité de la propriété de Painlevé pour trouver de nouveaux cas d'équations différentielles intégrables. De même, la conjecture d'Ablowitz, Ramani et Ségur permet de prouver l'intégrabilité de certaines classes d'équations différentielles (celles qui sont obtenues par réduction d'équations aux dérivées partielles intégrables par I.S.T.). De toutes ces approches est née une conjecture reliée directement aux équations différentielles ordinaires : Si une équation différentielle ordinaire possède la propriété de Painlevé, elle est intégrable.

Cette conjecture a permis de trouver de nombreux systèmes intégrables ; en particulier le système de Lorenz :

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = -y + rx - xz \\ \dot{z} = -bz + xy \end{cases}$$

a été étudié par Ségur [8].

Son analyse a donné quatre choix de paramètres σ, b, r, pour lesquels ce système possède la propriété de Painlevé :

i) $\sigma = 0$ (les équations sont linéaires)

ii) $\sigma = \frac{1}{2}$, $b = 1$, $\sigma = 0$

iii) $\sigma = 1$, $b = 2$, $r = \frac{1}{q}$

iv) $\sigma = \frac{1}{3}$, $b = 0$, r libre

Dans tous les cas, le système peut, soit être intégré par quadratures, soit être ramené à des correspondantes connues (PII).

De la même manière, de nombreux candidats de systèmes hamiltoniens intégrables ont été isolés grâce à cette méthode. Parmi eux, on peut citer l'hamiltonien de Hénon-Heiles, le système de Toda avec des masses inégales et des interactions de portées inégales que nous avons intégré récemment (voir chapitre 3).

En poussant plus loin la conjecture, on peut se demander si la propriété de Painlevé est nécessaire pour avoir intégrabilité, auquel cas elle fournirait un test exact pour l'intégrabilité.

Un exemple, fourni par un potentiel quintique bidimensionnel, montre que ce n'est pas le cas.

En effet, considérons le potentiel suivant :

$$V(x,y) = y^5 + y^3x^2 + \frac{3}{16} yx^4$$

Le système hamiltonien associé est intégrable. Il possède une seconde intégrale du mouvement qui vaut :

$$C = -y\dot{x}^2 + x\dot{x}\dot{y} + \frac{1}{2} x^2y^4 + \frac{3}{8} y^2x^4 + \frac{1}{32} x^6$$

Cependant, une analyse des singularités montre que les solutions x(t), y(t) ont un développement de la forme :

$$(t-t_0)^{-2/3} \sum_k a_k (t-t_0)^{k/3}$$

Elles ne vérifient donc pas la propriété de Painlevé bien que le système soit intégrable.

Ce cas nous a conduit à élargir la propriété de Painlevé de la manière suivante :

On dira qu'un système hamiltonien à deux dimensions possède la *propriété de Painlevé faible* (weak-Painlevé) si sa solution peut s'exprimer au voisinage de la singularité t_0 sous la forme de puissances de $(t-t_0)^{1/r}$, où r est un entier fixé par le comportement dominant de la singularité.

La nouvelle conjecture est alors que les systèmes hamiltoniens à deux dimensions, intégrables, ont la propriété de Painlevé.

La considération de cette propriété de Painlevé faible nous a permis de déterminer une famille d'hamiltoniens qui, satisfaisant à la propriété de Painlevé faible, se trouve en fait posséder une seconde intégrale du mouvement quadratique dans les vitesses. De plus, dans un système adéquat de coordonnées, l'équation pour les trajectoires se sépare effectivement. Il existe aussi des hamiltoniens intégrables non séparables qui possèdent la propriété de Painlevé faible, par exemple :

$$H_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x^2 - y^2)^{-2/3}$$

avec une seconde intégrale cubique dans les vitesses (due à Fokas [9]).

L'analyse Painlevé montre qu'il existe une solution avec un développement où toutes les puissances de $(t-t_0)^{1/5}$ sont présentes. Nous avons affaire à une réelle situation "Weak-Painlevé".

Il est de plus en plus probable que cette propriété pourra être associée à des systèmes dynamiques de dimensions supérieures. En particulier, la classe de tous les systèmes hamiltoniens, à N degrés de liberté, intégrables avec N intégrales de mouvement quadratiques dans les vitesses, a elle aussi été déterminée. Elle correspond à des potentiels séparables dans des systèmes de coordonnées adéquats. Il y a de fortes présomptions pour que, de manière analogue à ce qui se produit dans le cas de deux dimensions, on ait besoin de la propriété de Painlevé faible. Pour les systèmes avec un nombre de degrés de liberté infinie, la propriété de Weak-Painlevé semble aussi être la généralisation intéressante.

En effet, l'équation de Dym-Kruskal (qui s'obtient à partir de l'équation KdV à l'aide d'un changement de variable) a la forme suivante :

$$u_t = \frac{\partial^3}{\partial \phi^3}(u^{-1/2})$$

Cette équation est connue pour être intégrable [10].

Cependant, son développement autour de la variété des singularités $\Psi = 0$ est de la forme suivante :

$$V = \Psi^{2/3} \sum_{j=0}^{\infty} V_j \Psi^{j/3}$$

et les résonances sont à $j = -3, 2, 4$, ce qui est un comportement Painlevé faible typique.

Signalons pour terminer les travaux récents de H. Yoshida [11] qui se présentent comme une tentative de démonstration de la conjecture de Painlevé pour les O.D.E. Il considère des systèmes qui possèdent des propriétés de similarité, c'est-à-dire des systèmes de la forme :

$$\frac{dx_i}{dt} = F_i(x_1, \ldots, x_n) \qquad (F_i : \text{fonctions rationnelles}) \qquad (11)$$

tels que :

$$F_i(\alpha^{g_1}x_1,\ldots,\alpha^{g_n}x_n) = \alpha^{g_i+1}F_i(x_1,\ldots,x_n) \quad (\forall i = 1,\ldots,n)$$

où les g_j sont des nombres rationnels.

Son idée est la suivante :

Si le système *(11)* est algébriquement intégrable (c'est-à-dire qu'il existe k, $k \leqslant n-1$, constantes du mouvement algébriques et (n-k-1) autres qui s'expriment comme intégrales de formes différentielles de degré un), alors toutes les résonances du système sont *rationnelles*.

Même si son travail est encore incomplet, il est intéressant de noter que ce sont les rationnels (et non les entiers) qui interviennent de manière naturelle en tant que résonances. Ce travail renforce encore l'importance actuelle et à venir de la propriété de Painlevé faible dans l'étude de l'intégrabilité des systèmes dynamiques.

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Integrability in Dynamical Systems and the Painlevé Property

Bernadette Dorizzi

Centre National d'Etudes des Télécommunications
Issy les Moulineaux, France

Abstract

The analytic structure of the solution of an ordinary differential equation is intimately related to its integrability. The Painlevé property, *i.e.*, pure poles being the only movable singularities, allows the identification of new integrable dynamical systems. In this paper, we recall briefly the Ablowitz–Ramani–Segur (ARS) algorithm [4], which deals with an ordinary differential equation which possesses the Painlevé property, and applies it to hamiltonian dynamical systems with polynomial potentials of third and fourth order.

§1 Introduction

In this paper we shall focus on the study of integrable dynamical systems, the equations of motion of which are a system of ordinary differential equations. The search of integrability in such systems is difficult, but is rewarded by the nice behaviour of the trajectories. In fact, integrability allows long time predictions on the system. This is not possible when the only information available on the solution is given by numerical integration, or when the system exhibits a chaotic behaviour. In this last case, indeed, two arbitrarily close initial data give rise to rapidly diverging trajectories and, despite some recent progress based on renormalization and scaling, chaotic regimes remain unpredictable.

Unfortunately, it is not easy to detect integrability. The most widely used tools are numerical integration combined with observation of the section surfaces and group theoretical methods based on the study of dynamical symmetries. Substantial progress in this domain was made possible by the *Painlevé conjecture*. It is related to the analytic properties of the solutions of the equations of motion, namely, whenever the solutions possess the Painlevé property, *i.e.*, their only movable singularities on the complex-time plane are poles, the system is integrable. In order to clarify the concept of movable singularities, let us consider a *linear* second order differential equation:

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0.$$

Its general solution is:

$$x(t) = Ax_1(t) + Bx_2(t),$$

. Proceedings of the CIFMO School and Workshop
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where A and B are two arbitrary constants depending on the initial data, and $x_1(t)$ and $x_2(t)$ are two independent solutions. The location of the complex singularities of x does not depend on A and B, but on the singularities of p and q. They are fixed and independent of the initial values of the problem. Now, if the equation is *nonlinear*, a different kind of singularities can appear. If we consider the equation:

$$\dot{x} + x^2 = 0,$$

whose solution reads:

$$x(t) = \frac{1}{t - t_0},$$

we remark that the latter has a singularity at t_0, which is the constant of integration of the equation. This singularity is movable as it depends on the initial data of the problem. This conjecture has allowed the identification of the integrable, one-dimensional, ordinary differential equations of first and second order. At first order, the only Painlevé-type equations are the linear and the Ricatti ones. At second order, Painlevé and co-workers have identified 50 equations, 44 of which can be reduced to equations which are integrable in terms of elementary transcendental functions. The remaining six are new. Their solutions define the Painlevé trascendents. At higher orders, some partial classification exists (see [1]). In the case of the three-dimensional Lorenz system, Segur [2] has been able to identify the parameters for which the system is partially or fully integrable. Further cases of integrable systems with the Painlevé property have been considered by Bountis *et al.* in [3].

The remaining problem is thus to recognize, from the equation of motion, whether or not the system possesses the Painlevé property. A suitable algorithm has been developed by Ablowitz, Ramani, and Segur in [4]. We will first give a brief sketch of this method and illustrate it on an example. Then we will present some new integrable hamiltonian systems satisfying the Painlevé property.

§2 The Ablowitz–Ramani–Segur algorithm

We present here the method as developed in [4], for a single ordinary differential equation (ODE):

$$\dot{x} = f(x, t). \tag{2.1}$$

At the same time, we will illustrate it on the example of a system of two first order ODE's. At this point, one has to notice that this algorithm does not identify essential singularities and provides only necessary conditions for an equation to be of the Painlevé type. There are three steps in the algorithm:

Step 1. Find the dominant behaviour.
We assume that the solution becomes infinite at the singularity, and look for a solution of the form

$$x = \alpha(t - t_0)^p, \qquad \text{with } \operatorname{Re} p < 0, \tag{2.2}$$

which we substitute into (2.1).
For some values of p, some terms of the equations balance when $t \to t_0$, while the others can be ignored. They are called the *dominant terms* of the equations. If any of the possible p's is not an integer, the equation is not of the Painlevé type. Otherwise, one has to go on with the second step.

Example. Let us consider the system

$$\begin{cases} \dot{x} = x(a - x - y) & (2.3a) \\ \dot{y} = (x - 1)y & (2.3b) \end{cases}$$

We set

$$\begin{cases} x &= \alpha\tau^p \\ y &= \beta\tau^q, \end{cases} \qquad \tau = (t - t_0). \tag{2.4}$$

Substituting (2.4) into (2.3), we find the following values for p and q:

$$p = -1, \quad q = \alpha, \qquad \text{from (2.3b)},$$

and

$$\alpha = 1, \qquad \beta \text{ undetermined},$$

or

$$\alpha = -1, \qquad \beta = 2.$$

There are thus two distinct leading behaviours:

$$\begin{cases} x &= \tau^{-1} \\ y &= \beta\tau \end{cases} \quad \text{corresponding to the leading terms} \quad \begin{cases} \dot{x} &= -x^2 \\ \dot{y} &= xy, \end{cases} \tag{2.5}$$

$$\begin{cases} x &= -\tau^{-1} \\ y &= 2\tau^{-1} \end{cases} \quad \text{corresponding to the leading terms} \quad \begin{cases} \dot{x} &= -x^2 - xy \\ \dot{y} &= xy. \end{cases} \tag{2.6}$$

Step 2. Find the resonances.
For every negative integer value of p, the solution has an expression —at least formal— in the form of a Laurent series. The resonances are the powers of $(t - t_0)$ at which the different arbitrary constants enter in this expansion. To find them, one substitutes into the equations composed of the leading terms, the following form for x:

$$x = \alpha(t - t_0)^p + \beta(t - t_0)^{p+r}. \tag{2.7}$$

The equation then reduces to:

$$\mathcal{Q}(r)\beta(t - t_0)^q = 0. \tag{2.8}$$

The resonances are the roots of $\mathcal{Q}(r) = 0$. One can note that:

- one root is always -1 (arbitrariness of t_0),
- if α is arbitrary, 0 is also a root,
- a root with $\mathrm{Re}\, r > 0$, r noninteger, indicates a branch point,
- a root with $\mathrm{Re}\, r < 0$ is purely formal.

If there are no branch points, one has to go further with Step 3, dealing only with the positive integer resonances.

Example. We have two distinct cases of leading behaviour to consider:

Case 2.5: We set

$$\begin{cases} x &= \tau^{-1}(1 + \gamma\tau^r) \\ y &= \beta\tau(1 + \delta\tau^r). \end{cases} \tag{2.9}$$

We substitute (2.9) in

$$\begin{cases} \dot{x} &= -x^2 \\ \dot{y} &= xy, \end{cases}$$

and obtain the following system for γ and δ :

$$\begin{cases} \gamma(r+1) &= 0 \\ \gamma - \delta r &= 0. \end{cases} \tag{2.10}$$

It has nontrivial solution if and only if $r = -1$ or $r = 0$.

Case 2.6: We set

$$\begin{cases} x &= -\tau^{-1}(1+\gamma\tau^r) \\ y &= 2\tau^{-1}(1+\delta\tau^r). \end{cases} \tag{2.11}$$

We substitute (2.11) in

$$\begin{cases} \dot{x} &= -x^2 - xy \\ \dot{y} &= xy, \end{cases}$$

and obtain the following system for γ and δ :

$$\begin{cases} \gamma(1-r) - 2\delta &= 0 \\ 2\gamma + 2r\delta &= 0. \end{cases} \tag{2.12}$$

It has nontrivial solution if and only if $r = -1$ or $r = 2$.

Step 3. Find the constants of integration.
Let (p, α) given as in Step 1, $r_1 < r_2 < \cdots < r_s$, and the resonances, *i.e.*, the positive integer roots of $\mathcal{Q}(r)$, given from Step 2. We substitute

$$x = \alpha(t-t_0)^p + \sum_{j=0}^{s} \alpha_j (t-t_0)^{p+j}$$

into the full equation and calculate, by recurrence, the coefficients α_j. When $j = r_k$, $(k = 1, 2, \ldots, s)$, α_j is either undetermined or impossible to express. In the last case, one has to introduce *log* terms in the expansion and the equation is not of the Painlevé type.

Example. For the case (2.5) the resonances are -1 and 0, and the resulting Laurent series is thus algebraic. For the case (2.6), we must calculate the coefficients of the series up to the second order, *i.e.*, $r = 2$:

$$\begin{aligned} x &= -\tau^{-1} + a_1 + a_2\tau, & \dot{x} &= \tau^{-2} + a_2, \\ y &= 2\tau^{-1} + b_1 + b_2\tau, & \dot{y} &= -2\tau^{-2} + b_2. \end{aligned}$$

Equations (2.3) read:

$$\begin{aligned} \tau^{-2} + a_2 &= -(\tau^{-2} + a_1^2 - 2a_1\tau^{-1} - 2a_2) - (-2\tau^{-2} - b_1\tau^{-1} - b_2 + 2a_1\tau^{-1} + a_1b_1 + 2a_2) + (a\tau^{-1} + aa_1), \\ -2\tau^{-2} + b_2 &= -2\tau^{-2} - b_1\tau^{-1} - b_2 + 2a_1\tau^{-1} + a_1b_1 + 2a_2 - 2\tau^{-1} - b_1; \end{aligned}$$

then,

$$\begin{cases} b_1 &= a \\ a_1 &= \frac{1}{2}a + 1, \end{cases} \qquad \begin{cases} a_2 - b_2 &= aa_1 - a_1b_1 - a_1^2 \\ 2(a_2 - b_2) &= b_1 - a_1b_1. \end{cases}$$

The last system leads to the compalibility condition for $a_2 - b_2$: $2(-a_1^2) = b_1 - a_1b_1$, *i.e.*, $0 = 1 + a$. The system (2.3) is thus Painlevé if and only if $a = -1$.

§3 New cases of hamiltonian integrable systems

We will now consider the case of hamiltonian dynamical systems in two dimensions and show how the Painlevé conjecture allows the identification of new integrable cases. The hamiltonian has the form

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x,y), \tag{3.1}$$

and is a constant of the motion. Integrability, in such systems, is synonymous with the existence of a second constant of the motion, independent of the hamiltonian.

1. The Hénon–Heiles hamiltonian:

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + ax^2 + by^2) + dx^2y - \tfrac{1}{3}ey^3. \tag{3.2}$$

This hamiltonian has been studied by Chang *et al.* [5], and by Bountis and Segur [3]. The equations of motion are:

$$\begin{cases} \ddot{x} &= -ax - 2dxy \\ \ddot{y} &= -by - dx^2 + ey^2. \end{cases}$$

The Painlevé-analysis leads to two distinct leading behaviour (Step 1):

$$\begin{cases} x &= \alpha\tau^p \\ y &= \beta\tau^q, \end{cases} \quad \text{with} \quad \begin{matrix} p &= -2 \\ q &= -2, \end{matrix} \quad \begin{matrix} \alpha &= \pm(3/d)\sqrt{2+e/d} \\ \beta &= -3/d, \end{matrix}$$

and

$$\begin{matrix} p = \tfrac{1}{2} \mp \tfrac{1}{2}\sqrt{1-48d/e}, & \alpha \text{ arbitrary}, \\ q = -2, & \beta = 6/e. \end{matrix}$$

The resonances (Step 2) are:

$$n = -1,\ 6,\ \tfrac{5}{2},\ \pm\tfrac{1}{2}\sqrt{1-24(1+\tfrac{e}{d})},$$

or

$$n = -1,\ 0,\ 6,\ \pm\sqrt{1-48d/e}.$$

These values lead to integer values for the following values of the parameters:

$$\text{(a)} \begin{cases} e &= -d \\ a &= b, \end{cases} \qquad \text{(b) } e = -2d, \qquad \text{(c) } e = -6d, \qquad \text{(d)} \begin{cases} e &= -16d \\ b &= 16a. \end{cases}$$

In fact, case (c) is not a Painlevé case, as can be seen in performing Step 3 (one has to introduce *log* terms). In all other cases, the constants of motion have been calculated:

Case (a) $C = \dot{x}\dot{y} + axy + \frac{1}{3}dx^3 + dxy^2$. This case is separable in the $x+y$, $x-y$ coordinate system.

Case (b) $C = [(b-4a)+4dy]\dot{x}^2 - 4dx\dot{x}\dot{y} - d^2x^4 + a(b-4a)x^2 - 4dax^2y - 4d^3x^2y^2$. This integral is due to J. Greene [6].

Case (c) $C = 3\dot{x}^4 + 6(a+2dy)x^2\dot{x}^2 - 4dx^3\dot{x}\dot{y} - 4dx^4(ay+dy+3a^2x^4) - \frac{2}{3}d^2x^8$. This last constant is of fourth order in the velocities and has been found independently by Hall in [7], and by Grammaticos, Dorizzi, and Padjen in [8].

More generally, we have studied the case of hamiltonians with homogeneous polynomials potentials of third degree in [9].

2. Homogeneous third degree polynomial potentials:

$$V = y^3 + ay^2x + byx^2 + cx^3. \tag{3.3}$$

The coefficient a can be eliminated through a rotation, except for $(x+iy)^3$ and $(x^2+y^2)(x\pm iy)$, which are integrable cases. The Painlevé analysis is rather complicated in that case and the search for resonances leads to the following equation for the N_i's of the form:

$$36(N_1 + N_2 + N_3 - 12) = N_1\,N_2\,N_3\,, \tag{3.4}$$

where $N = m(m+1)$, with m an integer or half-integer. The solution $N_1 = 6$, $N_2 = 6$, N_3 free, is to be rejected. It corresponds to the potential $V = y^3 + \frac{3}{2}\,yx^2 + dx^3$, and leads to logarithmic singularities. The other solutions of (3.4), up to permutations of N_1, N_2, and N_3, which recovers $a = 0$, read

N_1	N_2	N_3	Potential	Order of the constant
0	0	12	$x^3 + y^3$	2^{nd}
2	30	30	$y^3 + \frac{1}{2}\,yx^2$	2^{nd}
$\frac{3}{4}$	90	90	$y^3 + \frac{3}{16}\,yx^2$	4^{th}
2	20	90	$y^3 + \frac{1}{2}\,yx^2 + i\,\frac{1}{6\sqrt{3}}\,x^3$	4^{th}

The three first cases recover the results of the Hénon–Heiles potential [with $a = b = 0$ in Eq. (3.2)], which we presented in paragraph one. The last case is a new one. The second constant is of degree four in the velocities and reads:

$$\begin{aligned} C = {} & \dot{y}^4 + 2\dot{y}^2\dot{x}^2 - i\,\tfrac{2}{\sqrt{3}}\,\dot{y}\,\dot{x}^3 + (4y^3 + 2yx^2 - i\,\tfrac{1}{3\sqrt{3}}\,x^3)\dot{y}^2 + (i\sqrt{3}\,yx^2 + x^3)\dot{x}\,\dot{y} \\ & + (4y^3 - 2i\sqrt{3}\,y^2x - i\,\tfrac{1}{3\sqrt{3}}\,x^3)\dot{x}^2 + 4y^6 + 4y^4x^2 + i\,\tfrac{1}{3\sqrt{3}}\,x^3y^3 + \tfrac{3}{4}\,y^2x^4 + i\,\tfrac{1}{6\sqrt{3}}\,x^5y + \frac{1}{54}\,x^6 \end{aligned} \tag{3.5}$$

This exhausts *all* the Painlevé cases for potentials of the form (3.3), and the direct approach for the computation of the integrals of motion allows the calculation of the constants in each case.

3. Homogeneous fourth degree polynomial potentials. We begin by introducing a potential V of the form:

$$V = y^4 + ay^3x + by^2x^2 + cyx^3 + dx^4. \tag{3.6}$$

As in the preceding case, the coefficient a can be set to zero by an adequate rotation, except for the potential

$$V = (x \pm iy)^4 + \mu(x^2 + y^2)(x \pm iy)^2,$$

which is indeed integrable. At this point, we do not pretend to be exhaustive and we will limit ourselves to potentials of the form:

$$V = y^4 + ay^2x^2 + bx^4,$$

which are parity-symmetric separately in x and in y. In this form, there are two well-known integrable cases:

Potential	Order of the constant
$x^4 + y^4$,	2^{nd}: (separable),
$(x^2 + y^2)^2$,	1^{st}: (the constant is just the angular momentum).

We have found two other new integrable cases:

$$y^4 + \tfrac{3}{4}x^2y^2 + \tfrac{1}{16}x^4, \qquad 2^{\text{nd}}\text{: } C = -y\dot{x}^2 + x\dot{x}\dot{y} + 8x^2y^3 + 4x^4y,$$

$$y^4 + \tfrac{3}{4}x^2y^2 + \tfrac{1}{8}x^4, \qquad 4^{\text{th}}\text{: } C = \dot{x}^4 + (24x^2y^2 + 4x^4)\dot{x}^2 - 16x^3y\dot{x}\dot{y} + 4x^4\dot{y}^2 + 4x^8 + 16x^6y^2 + 16x^4y^4.$$

§4 Conclusion

In this paper, we have shown how the complex-plane singularity analysis of the equations of motion is interesting for the study of integrability of ODE's. The Painlevé analysis has led us to the discovery of new integrable dynamical systems. They correspond to the motion of a particle in two-dimensional homogeneous polynomial potentials of third and fourth degree. In the case of cubic interaction, we have been able to identify *all* the Painlevé cases and to calculate the second constant of motion for each of them. For the case of quartic interaction, due to the complexity of the problem, our search was not exhaustive but we nevertheless discovered new integrable potentials.

The Painlevé conjecture thus appears to be a powerful tool in the study of dynamical systems, and it is far from having shown the limits of its possibilities.

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CHAPITRE 5

SYSTÈMES HAMILTONIENS

Cette partie sera consacrée à la présentation de travaux que nous avons effectués sur les systèmes hamiltoniens autonomes.

De tels systèmes sont caractérisés par l'existence d'une fonction H qui dépend des positions q_i et des impulsions p_i ; i varie de 1 à N où N désigne le degré de liberté du système. Cette fonction H permet de décrire l'évolution du système puisque les équations du mouvement ont la forme suivante :

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$(i = 1,\ldots,N) \qquad (1)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Nous nous sommes intéressés à l'intégrabilité de tels systèmes.

A priori, on a affaire à un système de 2N équations différentielles ordinaires, autonomes qui possède toujours l'hamiltonien H comme constante du mouvement. Si on connaît en tout 2N-1 intégrales premières indépendantes d'un tel système, on peut au moins implicitement ramener sa résolution à une quadrature.

Dans ce cas, le système peut être considéré comme complètement intégrable. Néanmoins, du fait de la structure très particulière des systèmes hamiltoniens, nous allons voir que la connaissance de N constantes du mouvement permet d'assurer l'intégrabilité au sens précédent.

Plus précisément, supposons qu'il existe N intégrales premières, monovaluées, indépendantes, analytiques et en involution (c'est-à-dire de crochets de Poisson nul), $\phi_1 = H, \phi_2 \ldots, \phi_N$. On peut alors construire N-1 autres intégrales premières $\Omega_1,\Omega_2,\ldots,\Omega_{N-1}$ de la manière suivante :

On introduit la transformation canonique :

$$(q_i,p_i) \rightarrow (q_i^*,p_i^*)$$

définie à l'aide d'une fonction génératrice $S(q,p^*)$ par :

$$p_i = \frac{\partial S}{\partial q_i}$$

$$q_i = \frac{\partial S}{\partial p_i^*}$$

de manière à ce que le nouveau moment p_i^* soit égal à $\phi_i(q,p)$.

La fonction S est définie par :

$$S = \int \sum_{i=1}^{N} p_i(q,p^*)dq_i \qquad (2)$$

où $p_i(q,p^*)$ est la solution des équations :

$$\phi_1(q,p) = p_1^*$$
$$\phi_N(q,p) = p_N^*$$

C'est parce que les ϕ_i sont en involution que l'intégrale *(2)* a bien un sens.

Dans le nouveau système de coordonnées, les équations de mouvement prennent la forme suivante :

$$\frac{dq_i^*}{dt} = \frac{\partial H^*}{\partial p_i^*}$$

$$\frac{dp_i^*}{dt} = -\frac{\partial H^*}{\partial q_i^*}$$

avec $H^*(q^*,p^*) = H(q(q^*,p^*), p(q^*,p^*)) = p_1$.

Ce système s'intègre immédiatement pour donner :

$$p_i^* = \alpha_i \quad \text{(constante)} \quad (i = 1,2,\ldots,N)$$

$$q^* = t + \text{cste}, \quad q_j^* = \beta_j = \text{cste} \quad (j = 1,\ldots,N)$$

Ainsi, en plus des N premières intégrales :

$$\phi_1 = p_1^* = \alpha_1, \ \ldots, \ \phi_N = p_N^* = \alpha_N,$$

il existe N-1 intégrales premières, indépendantes du temps, $\Omega_i = q_i^*$ $(i = 1,2,\ldots,N)$, en général *non monovaluées*.

En effet, les Ω_i s'expriment comme intégrales de la forme différentielle suivante de degré 1 :

$$\Omega_i = \sum_{j=1}^{N} \int_{q_{j,0}}^{q_j} \phi_{i,j}(q,p^*)dq_j \qquad (i = 1,\ldots,N)$$

Les $\phi_{i,j}(q,p^*)$ se construisent de manière algébrique à partir des dérivées partielles de $\phi_1,\ldots,\phi_N$.

A la vue de ce résultat, il paraît naturel de dire, suivant une définition communément admise, qu'un système hamiltonien de dimension N est intégrable lorsqu'il possède N intégrales premières, analytiques, indépendantes et en involution.

Nous avons surtout travaillé en dimension N = 2 à partir d'hamiltoniens de la forme :

$$H = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + V(x,y) \qquad (3)$$

Ces systèmes sont plus faciles à étudier. En effet, étant donné que la dimension est 2, il suffit pour prouver l'intégrabilité d'exhiber une seconde constante du mouvement indépendante de l'hamiltonien. De plus, on peut se restreindre à chercher cette constante polynomiale dans les vitesses, comme l'hamiltonien H.

On peut alors chercher à déterminer systématiquement de telles constantes, ordre par ordre ; il suffit pour cela d'écrire l'expression générale d'un polynôme de degré k en $(\dot{x},\dot{y})$. Seuls les termes de même parité sont présents étant donné que l'hamiltonien est invariant par renversement du temps.

Une telle constante va s'écrire :

$$C = \sum_{i=0}^{k} f_i^k \dot{x}^{k-i}\dot{y}^i + \sum_{i=0}^{k-2} g_i^{k-2} \dot{x}^{k-2-i}\dot{y}^i + \ldots$$

Les f_i^k, g_i^k, etc. sont des fonctions de x et y à déterminer.

Si dans $\frac{dC}{dt} = 0$, on remplace $\ddot{x}$ et $\ddot{y}$ par leurs expressions en fonction du potentiel V, on est en présence d'un polynôme en $(\dot{x},\dot{y})$ qui doit être identiquement nul pour que C soit une constante du mouvement.

L'annulation des coefficients de chaque monôme de $\frac{dC}{dt}$ conduit à une série d'équations aux dérivées partielles à satisfaire par les fonctions f_i, g_i, h et V.

Pour que ces équations soient solubles, certaines relations de compatibilité devront être satisfaites. Elles imposent des contraintes sur la forme du potentiel V, ainsi que sur les fonctions f_i, g_i et h.

Si, par exemple, le potentiel V dépend d'un paramètre continu λ, cette méthode permettra de sélectionner éventuellement certaines valeurs de ce paramètre. La suite du calcul, par la détermination des fonctions f_i^k, g_i^k permettra alors d'exhiber explicitement la constante du mouvement.

Ainsi, théoriquement, si on cherche à intégrer un système hamiltonien donné de la forme *(3)*, il suffit de chercher ordre par ordre une constante polynomiale dans les vitesses. Malheureusement, les calculs deviennent de plus en plus compliqués au fur et à mesure que le degré du

polynôme augmente. Au-delà de l'ordre 6, de tels calculs semblent impraticables à la main. C'est pourquoi, en pratique, on utilise des détecteurs qui ne remplacent pas le calcul explicite des constantes du mouvement, mais permettent au moins de sélectionner les cas potentiellement intégrables. Dans notre travail, c'est le critère de Painlevé et la théorie des algèbres de Lie qui ont permis de sélectionner des candidats d'intégrabilité ; candidats qui ont en plus été explicitement intégrés.

Dans la suite de cette partie, c'est surtout la propriété de Painlevé qui a servi de détecteur. En fait, il y a dans nos travaux une interpénétration constante entre la méthode de recherche directe des constantes du mouvement et l'utilisation du critère de Painlevé.

En effet, nous nous sommes d'abord intéressés au potentiel de Hénon-Heiles :

$$V = x^2y - \frac{e}{3}y^3 + x^2 + ay^2$$

On vérifie que, pour e = -6, e = 1 et a = 1, e = -16 et a = 16 , le système possède la propriété de Painlevé. Une deuxième intégrale du mouvement était déjà connue pour les deux premiers cas. Dans la première publication présentée, nous donnons la constante du dernier cas.

D'un autre point de vue, la méthode de recherche systématique des constantes du mouvement nous a permis de trouver un potentiel quintique intégrable qui ne possède pas la propriété de Painlevé.

Nous avons ainsi été conduits à étendre la propriété de Painlevé de manière à pouvoir rendre compte de ce phénomène.

Les trois publications suivantes introduisent et utilisent cette nouvelle notion de "Painlevé-faible", qui nous a permis, entre autres, de déterminer la classe de tous les potentiels qui possèdent un second invariant quadratique dans les vitesses. Elle a aussi conduit à l'identification des potentiels cubiques et quartiques intégrables. Tous ces potentiels ont de plus été intégrés par la méthode directe.

Enfin, deux publications viennent compléter et enrichir cette notion de "Painlevé-faible" et sont des réponses à des questions précises concernant la conjecture. La première nous a été posée par M.D. Kruskal :

Serait-il possible de retrouver la propriété de Painlevé originelle par un changement de variable adéquat ?
Nous montrons que ce n'est pas le cas pour une classe de systèmes intégrables pour lesquels les équations de mouvement peuvent être intégrées par quadrature.

Une autre question de A. Fokas était de savoir si il existait des cas de systèmes intégrables, avec la propriété de Painlevé faible qui ne soient ni séparables, ni réductibles à des quadratures à l'aide d'un changement de variable adéquat. Nous avons alors étudié, de ce point de vue, deux potentiels qui possèdent une constante de degré trois dans les vitesses. Cette étude nous a permis de répondre positivement à la question et aussi d'exhiber de nouveaux cas intégrables avec des constantes d'ordre plus élevé dans les vitesses.

Volume 89A, number 3 PHYSICS LETTERS 3 May 1982

PAINLEVE PROPERTY AND INTEGRALS OF MOTION FOR THE HENON–HEILES SYSTEM

B. GRAMMATICOS, B. DORIZZI and R. PADJEN
Département de Mathématiques (MTI), Centre National d'Etudes de Télécommunications, 92131 Issy les Moulineaux, France

Received 18 February 1982

Using the Painlevé criterion three integrable cases are identified for the Hénon–Heiles system. A new integral of motion, unknown to date is presented for the third case.

One of the main features of dynamical systems is their non-integrability. A very small number of integrable dynamical systems is known to date and moreover, until very recently, no systematic procedure existed for the identification of an integrable system. However recently Ablowitz et al. [1] have proposed a criterion for integrability. This method is based on the Painlevé property of the solutions [2,3], i.e. the solutions possess movable singularities in the complex-time plane, which are just poles. It was conjectured thus [4] that a system which presents the Painlevé property is integrable. This applies to hamiltonian as well as to dissipative systems. Integrability for hamiltonian systems of N degrees of freedom implies the existence of N global analytical integrals which are time independent. In the case of the Lorenz system Segur [5] has been able to identify the parameters for which the system was partially or fully integrable. Tabor and Weiss [6] have presented a detailed analysis of the same system paying particular attention to the analytic properties of the solution and have corroborated the results of Segur. Further cases of integrable systems with the Painlevé property have been considered by Bountis et al. [7].

In this letter we focus on the Hénon–Heiles [8] hamiltonian system. It has been studied in particular by Chang and co-workers in refs. [9,10] where they have shown the existence and displayed the structure of a natural boundary for the solution for the non-integrable regimes. The investigation of the Painlevé property allows the identification of three integrable cases. While the integrals of motion on the first two cases are known [11,12], this appears not to be true for the third case. The main result of this letter is to present this new integral for the Hénon–Heiles system.

We start with a general expression of the Hénon–Heiles hamiltonian:

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + ax^2 + by^2) + dx^2y - \tfrac{1}{3}ey^3 \,. \tag{1}$$

The corresponding equations of motion are:

$$\ddot{x} = -ax - 2dxy \,,$$

$$\ddot{y} = -by - dx^2 + ey^2 \,. \tag{2}$$

In order to investigate the leading order behaviour around a singularity in the complex-time plane we put:

$$x = A/\tau^{\mu} \,, \quad y = B/\tau^{\nu} \,, \tag{3}$$

where $\tau = t - t_p$ with t_p the position of the complex pole.

Equating the dominant terms we obtain:

$$A\,\mu(\mu+1)/\tau^{\mu+2} = -2dAB/\tau^{\mu+\nu} \,,$$

$$B\,\nu(\nu+1)/\tau^{\nu+2} = eB^2/\tau^{2\nu} - dA^2/\tau^{2\mu} \,. \tag{4}$$

So two solutions are possible. Following Chang et al. we call *case 1* the solution:

$$\mu = 2 \,, \quad A = \pm(3/d)\sqrt{2 + e/d} \,,$$

$$\nu = 2 \,, \quad B = -3/d \,, \tag{5}$$

and *case 2* the solution:

$$\mu = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 48\,d/e} \,, \quad A \text{ arbitrary} \,,$$

$$\nu = 2 \,, \qquad B = 6/e \,. \tag{6}$$

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The Painlevé criterion requires that μ and ν be integers, which severely restricts the values of d/e in the case 2 singularities. However this is not sufficient in order to establish integrability. Following Ablowitz et al. [1,4] we investigate the so called resonances. We are looking namely for higher order terms of the form:

$$x = A/\tau^{\mu} + C/\tau^{\mu-r} ,$$

$$y = B/\tau^{\nu} + D/\tau^{\nu-r} , \qquad (7)$$

where r is determined by the requirement that C, D be arbitrary when the solution (7) is substituted back into the dominant part of the equations of motion and a linearization with respect to C and D is performed.

The analysis, somewhat lengthy but quite straightforward, gives for case 1 singularities:

$$r = -1, 6, \tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{1 - 24(1 + e/d)}$$

and for case 2

$$r = -1, 0, 6, \pm\sqrt{1 - 48d/e} . \qquad (8)$$

As explained in ref. [1] the root -1 is associated with the arbitrariness of the pole position while the root 0 in case 2 is related to the arbitrariness of A. For the system to be integrable according to the Painlevé criterion the values of r must be integers. This turns out to be the case for

$$e = -6d , \qquad e = -2d , \qquad e = -d . \qquad (9)$$

The case $e = -d$ correspond to the well-known [11] separable case of the Hénon–Heiles system. The case $e = -2d$ does not lead to integrability as explained by Chang et al. [9]. The case $e = -6d$ is a new integrable case and Greene [12] has obtained the corresponding integral of motion ($d = 1$):

$$G = x^4 + 4x^2y^2 - 4\dot{x}(\dot{x}y - \dot{y}x) + 4ax^2y + (4a - b)(\dot{x}^2 + ax^2) . \qquad (10)$$

In this case the integrability was actually deduced from the solution based on case 2 singularities. This suggests further possibilities; namely the system may be integrable whenever the Painlevé criterion is satisfied for the case 2 resonances only. Chang et al. [10] (see also Bountis and Segur [13]) propose thus a new candidate for integrability namely the parameter values:

$$e = -16d , \qquad b = 16a , \qquad (11)$$

for which numerical evidence of integrability exists as well. We proceed now to demonstrate that this case is indeed integrable and calculate this new integral of motion.

We look for an integral of motion which contains velocities up to the fourth power (no odd powers due to time reversal invariance),

$$C = f_1\dot{x}^4 + f_2\dot{x}^3\dot{y} + f_3\dot{x}^2\dot{y}^2 + f_4\dot{x}\dot{y}^3 + f_5\dot{y}^4 + g_1\dot{x}^2 + g_2\dot{x}\dot{y} + g_3\dot{y}^2 + h , \qquad (12)$$

where f_i, g_i, h are general functions of x and y to be determined. Using the energy integral we can eliminate $\dot{y}^2, \dot{y}^4$ from (12), readjusting accordingly the remaining functions f_i, g_i and h. We are thus left with

$$C = f_1\dot{x}^4 + f_2\dot{x}^3\dot{y} + g_1\dot{x}^2 + g_2\dot{x}\dot{y} + h . \qquad (13)$$

A remark seems in order at this point. The procedure we outline here, is not a heuristic one. A general form of the integral of motion up to a $2N$th order in the velocities can be given

$$C = \sum_{n=1}^{N} f_1^n\dot{x}^{2n} + f_2^n\dot{x}^{2n-1}\dot{y} + h$$

and systematically explored. However as it will turn out, form (13) will suffice in our case.

We write the constant of motion condition $dC/dt = 0$ and equate the coefficients of the various powers of the velocities to zero. To first order we get

$$\partial f_1/\partial x = 0 ,$$

$$\partial f_1/\partial y + \partial f_2/\partial x = 0 ,$$

$$\partial f_2/\partial y = 0 . \qquad (14)$$

The solution of which is

$$f_1 = \alpha y + \beta , \qquad f_2 = -\alpha x + \gamma . \qquad (15)$$

To third order the result is

$$f_2\dddot{y} + 4f_1\dddot{x} + \partial g_1/\partial x = 0 ,$$

$$3f_2\ddot{x} + \partial g_1/\partial y + \partial g_2/\partial x = 0 ,$$

$$\partial g_2/\partial y = 0 . \qquad (16)$$

The last of equations (16) gives $g_2 = g_2(x)$. From the two remaining equations (16) we get:

$$-\frac{\partial^2 g_1}{\partial x\,\partial y} = \frac{\partial}{\partial y}(f_2\dddot{y} + 4f_1\dddot{x}) = \frac{\partial}{\partial x}\left(3f_2\ddot{x} + \frac{dg_2}{dx}\right) \qquad (17)$$

and

$$\frac{d^2g_2}{dx^2} = \frac{\partial}{\partial y}(f_2\ddot{y} + 4f_1\ddot{x}) - \frac{\partial}{\partial x}(3f_2\ddot{x}) . \qquad (18)$$

This means that the second member must be independent of y. This leads to the following conditions:

$$e = -6d \quad \text{or} \quad \alpha = 0 ,$$

$$e = -3d \quad \text{or} \quad \gamma = 0 . \qquad (19)$$

If we stick to the $e = -16d$ condition we obtain: $\alpha = \gamma = 0$, and thus $f_1 = \beta, f_2 = 0$. Integrating equations (16) we obtain:

$$g_1 = 2\beta(a + 2dy)x^2 + c ,$$

$$g_2 = -\tfrac{4}{3}\beta dx^3 + k , \qquad (20)$$

where c, k are two integration constants.

We proceed now to first order:

$$2g_1\ddot{x} + g_2\ddot{y} + \partial h/\partial x = 0 ,$$

$$g_2\ddot{x} + \partial h/\partial y = 0 . \qquad (21)$$

The compatibility condition is

$$-\frac{\partial^2 h}{\partial x\,\partial y} = \frac{\partial}{\partial y}(2g_1\ddot{x} + g_2\ddot{y}) = \frac{\partial}{\partial x}(g_2\ddot{x}) \qquad (22)$$

which amounts to

$$e = -16d \quad \text{or} \quad \beta = 0 ,$$

$$b = 16a \quad \text{or} \quad \beta = 0 ,$$

$$c = 0 ,$$

$$k = 0 . \qquad (23)$$

We remark thus that the conditions for the existence of the integral of motion are identical to the ones (11) found by Chang et al. [10].

Integration of (21) gives for h:

$$h = -\tfrac{4}{3}\beta dx^4(ay + dy^2) + \beta a^2x^4 - \tfrac{2}{9}\beta d^2x^6 \qquad (24)$$

and finally the integral of motion becomes (with $\beta = 3$ for the multiplicative constant):

$$C = 3\dot{x}^4 + 6(a + 2dy)x^2\dot{x}^2 - 4dx^3\dot{x}\dot{y} -$$

$$- 4dx^4(ay + dy^2) + 3a^2x^4 - \tfrac{2}{3}d^2x^6 . \qquad (25)$$

This integral together with the energy integral (1) allows in principle the integration of the Hénon–Heiles system for $e = -16d$ and $b = 16a$.

In their paper Chang et al. [10] suggest that other cases of integrability of the Hénon–Heiles system may exist for values of the ratio d/e for which $\sqrt{1 - 48d/e} = n/m$ with n and m two relatively prime integers (based upon the case 2 singularities analysis). Our method for the systematic construction of the integral of motion may turn out to be a useful tool for the investigation of such integrability candidates.

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PHYSICAL REVIEW
LETTERS

VOLUME 49 22 NOVEMBER 1982 NUMBER 21

Painlevé Conjecture Revisited

A. Ramani
Centre de Physique Theorique, Ecole Polytechnique, F-91128 Palaiseau, France

and

B. Dorizzi and B. Grammaticos
Département de Mathématiques (M.T.I.), Centre National d'Etudes de Télécommunications, F-92131 Issy les Moulineaux, France

(Received 26 April 1982)

The discovery of new integrable two-dimensional Hamiltonian systems is reported. The analytic structure of the solutions makes necessary the generalization of the Painlevé conjecture, a widely used integrability criterion. Such a generalization is presented, which the authors believe should replace the usual conjecture for two-dimensional Hamiltonian systems. It is indeed compatible with all the systems already found and, in addition, leads to still new integrable cases.

PACS numbers: 03.20.+i, 02.30.+g

The extreme rarity of integrable dynamical systems makes the quest for them all the more exciting. Till very recently, the search for such systems was based mainly on intuition and on extensive (and expensive!) numerical investigations. However, in the last few years the works of Ablowitz, Ramani, and Segur (ARS)[1-3] have provided us with a most useful tool for the identification of integrable systems. They have conjectured, and, subsequently abundantly verified, that a dynamical system described by a differential system or a partial differential equation is integrable whenever the solutions have the Painlevé property,[4] namely, that their only movable singularities are poles.

The method has met with an undeniable success as it has guided the search for integrals of motion for dynamical systems of physical interest. In the case of the Lorenz system, Segur was able to identify integrable cases with the help of the Painlevé property and proceeded to calculate the integrals of motion.[5] The Henon-Heiles Hamiltonian is another example. The analysis by Chang *et al.* has resulted in three cases having the Painlevé property.[6] One corresponds to a well-known separable case.[7] In the second case the integral of motion was calculated by Greene.[8] The recent derivation of the second integral for the third case independently by Hall[9] and Grammaticos, Dorizzi, and Padjen[10] fully confirmed the predictive character of the Painlevé property.

At this point one can ask a most pertinent question. Do all integrable systems have the Painlevé property? This has been tacitly assumed and actually verified on a host of integrable cases to date.[6,11,12] However, some recent results, if interpreted in a narrow-minded way, would tend to suggest the contrary. The aim of this paper is to present these new integrable systems and to show how the conjecture concerning the Painlevé

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property must be extended so as to accomodate these new cases of integrability. The need to generalize the ARS conjecture to allow for a class of singularities less restrictive than was previously thought can be seen from the following remarks:

(a) One-dimensional Hamiltonian systems are integrable by definition. Clearly, the integrability is independent of the analytical properties of the solution in the complex time plane. Logarithmic cuts or even essential singularities which can easily be built into the Hamiltonian do not hinder integrability. (b) It has long been known that the Painlevé property should be extended in order to include a global change of variables, provided this single change transforms the singularities of *all* the solutions into poles.[3] This was considered a minor generalization of the Painlevé property. However, the conclusive case was the discovery of (c) an integrable Hamiltonian system in two dimensions, for a mass-one particle, with a quintic polynomial potential:

$$V = y^5 + y^3x^2 + \tfrac{3}{16}yx^4.$$

This system has a second integral of motion in addition to the total energy. This integral is

$$C = -y\dot{x}^2 + x\dot{x}\dot{y} + \tfrac{1}{2}x^2y^4 + \tfrac{3}{8}y^2x^4 + \tfrac{1}{32}x^6.$$

An analysis of the complex time (t) singularities of the system shows that they start as $(t-t_0)^{-2/3}$. This, in itself, need not be a problem. It might, in principle, be accomodated by remark (b) above, had the cube of the solution had a pure double pole. However, the expansion of the solution around the singularity contains *all* powers of $(t-t_0)^{1/3}$. This is in direct contradiction with the ARS conjecture as previously meant and leads us to accept a wider class of movable singularities.

In order to find a new integrability criterion, we start by excluding one-dimensional systems and fully separable cases which are equivalent to them. Analytic properties do not play any role in these cases. For two-dimensional systems, we will say that a system possesses the "weak Painlevé property" whenever the solution, in the neighborhood of a singularity at t_0, can be expressed as an expansion in powers of $(t-t_0)^{1/r}$. Here r is an integer which must be the "natural" one, given solely by the leading behavior of the singularity. The new conjecture is now that integrable systems have the weak Painlevé property. At this point we emphasize that one can recover poles neither by raising the solution to the rth power, nor by a change of independent variable, since the singularity is movable. This is a genuine weakening of the Painlevé property. However singularities of logarithmic or irrational power type are still conjectured to be incompatible with integrability. For example,

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + ax^3 + by^3 + cxy$$

has a logarithmic singularity for $c \neq 0$ and is only integrable for $c = 0$.

For the more than two-dimensional cases, one can only surmise that the need for the weakening of the Painlevé concept will become less and less felt with increasing dimensionality. Already in three dimensions there does not seem to be any counterexample to the original Painlevé conjecture. However, results in these cases are scant and any systematic search prohibitively involved. For infinite dimensions, that is, partial differential equations, it seems safe to put forward that the conjecture will remain valid in its original form.

Now that the conjecture has been reformulated, one must question its predictive character. Can one use it to find new integrable systems? The following will show that the answer to this question is affirmative.

A first example is offered by a family of Hamiltonians to which the example in (c) belongs. These Hamiltonians describe a mass-one particle in two dimensions subject to a homogeneous polynomial potential of the form

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} C_{n-k}{}^k x^{2k} y^{n-2k}.$$

One can actually show that all these Hamiltonians are integrable, the second integral of motion being

$$C_n = -y\dot{x}^2 + x\dot{x}\dot{y} + \sum_{k=0}^{[(n-1)/2]} 2^{n-2k-1} C_{n-k-1}{}^k x^{2k+2} y^{n-2k-1},$$

that is,

$$C_n = -y\dot{x}^2 + x\dot{x}\dot{y} + x^2 V_{n-1}.$$

One has found this family of homogeneous polynomials by attempting to satisfy the weak Painlevé property in the easiest way through some simplifying assumptions. This leads to the following partial differential equation for V:

$$2y\frac{\partial^2 V}{\partial x \partial y} + 3\frac{\partial V}{\partial x} + x\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}\right) = 0.$$

The same partial differential equation is obtained if one looks directly for an integral of motion quadratic in the velocities. This shows that any linear combination of the V_n's also leads to an integrable Hamiltonian. Actually the Henon-Heiles potential, in the case integrated by Greene,[8] obeys just this equation, and is a linear combination of V_3, V_2, and V_1, up to a suitable linear transformation of variables. The derivation of this equation, together with its general solution, which we have obtained, will be presented in a future publication.

The family of integrable potentials we have just presented are not the only ones. The class of integrable systems can be further expanded if one relaxes the simplifying assumptions we alluded to previously. However, the calculations become hopelessly intricate unless one limits oneself to some low-order polynomial potentials.

We have fully investigated the case of cubic polynomial potentials, and have been able to find *all* weak Painlevé cases. These include all the known integrable cases[8,9] and, in addition, a unique (up to rotation and complex conjugation) new case,

$$V = y^3 + \tfrac{1}{2}yx^2 + ix^3/6\sqrt{3},$$

for which the integral has been found:

$$C = \dot{y}^4 + 2\dot{y}^2\dot{x}^2 - 2i\dot{y}\dot{x}^3/\sqrt{3} + (4y^3 + 2yx^2 - ix^3/3\sqrt{3})\dot{y}^2 + (i\sqrt{3}yx^2 + x^3)\dot{x}\dot{y} + (4y^3 - 2i\sqrt{3}y^2x - ix^3/3\sqrt{3})\dot{x}^2 + 4y^6 + 4y^4x^2 + ix^3y^3/3\sqrt{3} + 5y^2x^4/4 + ix^5y/6\sqrt{3} + x^6/54.$$

One can show directly that this exhausts all third-order potentials allowing integrals which are polynomial of order four or less in the velocities.

At order four, we found two new integrable potentials by looking for the weak Painlevé property. One is the $n = 4$ case of the family described above,

$$V_4 = 16y^4 + 12x^2y^2 + x^4,$$

for which the integral is

$$C_4 = -y\dot{x}^2 + x\dot{x}\dot{y} + 8x^2y^3 + 4x^4y.$$

The other is

$$V = 8y^4 + 6x^2y^2 + x^4,$$

for which the integral is

$$C = \dot{x}^4 + (24x^2y^2 + 4x^4)\dot{x}^2 - 16x^3y\dot{x}\dot{y} + 4x^4\dot{y}^2 + 4x^8 + 16x^6y^2 + 16x^4y^4.$$

We do not claim that this exhausts the weak Painlevé, nor integrable, cases for fourth-order potentials.

We therefore conjecture that, for two-dimensional Hamiltonian systems, the appropriate generalization of the Painlevé property we introduced, rather than the original form of this property, is characteristic of integrability. Moreover, our results constitute ample evidence that this generalized Painlevé property is a most useful tool in the search for integrable dynamical systems.

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A new class of integrable systems[a)]

B. Dorizzi and B. Grammaticos
Departement de Mathématiques (M.T.I.), Centre National d'Etudes de Télécommunications, 92131 Issy les Moulineaux, France

A. Ramani
C. P. Th., Ecole Polytechnique, 91128 Palaiseau Cedex, France

(Received 12 October 1982; accepted for publication 17 December 1982)

We present a family of dynamical systems associated with the motion of a particle in two space dimensions. These systems possess a second integral of motion quadratic in velocities (apart from the Hamiltonian) and are thus completely integrable. They were found through the derivation and subsequent resolution of the integrability condition in the form of a partial differential equation (PDE) for the potential. A most important point is that the same PDE was derived through considerations on the analytic structure of the singularities of the solutions ("weak-Painlevé property").

PACS numbers: 02.30.Jr

I. INTRODUCTION

The aim of this paper is to present a "hidden treasure" class of integrable systems in two space dimensions. Given the extreme rarity of integrable dynamical systems, which has spurred an intense research in this domain, and the fact that at least one example of integrable system belonging to the aforementioned class was known since the beginning of celestial mechanics, it is astonishing that these systems have lain undiscovered for over 80 years.

In a 1901 paper, Darboux,[1] applying a method due to Bertrand,[2] obtained the general partial differential (PDE) which the potential must satisfy in order for the system to possess an integral of motion quadratic in the velocities. He then proceeded to specify and solve the equation in the general case, while being aware of the possibility of the existence of a particular case of integrability, the importance of which he did not grasp at the moment. The incomplete former case of the Darboux integrability was subsequently presented by Whittaker[3] as "the only case of the motion of a particle in a plane under the action of conservative forces which possess an integral quadratic in velocities other than the integral of energy." However, the recent discovery by Greene[4] of an integral of motion quadratic in the velocities for the Henon–Heiles system, which can easily be shown not to belong to the Darboux family, weakened the generality of the Darboux solution. Actually, the most classical example of a non-Darboux integral of motion is the third integral of motion for the Kepler problem, which has been known for several centuries.

It must be stressed at this point that integrability in our case of two-dimensional systems means the existence of a second integral of motion besides the energy of the system. However, we explicitly demand that this second integral exist for every value of the energy. An interesting generalization of the concept of integrability was obtained by Hall,[5] by releasing this constraint. In his approach, one can also obtain integrability for a larger class of systems, but only for some specific values of the energy.

A first result of our work was presented elsewhere.[6] It concerned a two-dimensional quintic homogeneous polynomial potential which can be shown to lead to an integrable dynamical system. The particularity of this potential was that the associated equations of motion did not possess the Painlevé property, which has been conjectured, and amply verified, by Ablowitz, Ramani, and Segur,[7] to be associated with integrability. This entailed a weakening of the Painlevé criterion for two-dimensional systems. We then proposed to replace it by a "weak Painlevé property," which is not just an artifice which allows us to explain the particularities of the quintic potential; it has a real predictive power. In the case at hand this property, suitable applied, allows us to derive a PDE to be obeyed by the potential (Sec. III). This equation is precisely the particular case of the PDE discovered by Darboux that the latter did not investigate (Sec. II). In Sec. IV we present the general solution of this PDE, and we display some cases of particular interest.

II. INTEGRALS OF MOTION QUADRATIC IN VELOCITIES

Let us consider the motion of a particle of unit mass in a two-dimensional potential $V(x,y)$. The Hamiltonian governing the system reads

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x,y). \tag{1}$$

The equations of motion associated with the system are simply

$$\ddot{x} = -\frac{\partial V}{\partial x} \equiv -V_x, \quad \ddot{y} = -\frac{\partial V}{\partial y} \equiv -V_y. \tag{2}$$

For the complete integrability of the system, one needs the existence of a second constant of motion, besides the Hamiltonian itself. In this paper we will focus on integrals of motion quadratic in the velocities, although integrals of order higher than two are not unknown.[6,8,9] The general form of such an integral is

$$C = g^{(0)}\dot{x}^2 + g^{(1)}\dot{x}\dot{y} + g^{(2)}\dot{y}^2 + h, \tag{3}$$

[a)] The authors dedicate this work to their Soviet colleague, the mathematician and physicist Nahum Meiman.

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where g and h are functions of x and y. No term linear in velocities is allowed as the Hamiltonian (1) is invariant under time reversal. The conditions of the constancy of C can be written as

$$0\equiv\frac{dC}{dt}=g_x^{(0)}\dot{x}^3+g_y^{(0)}\dot{x}^2\dot{y}+g_x^{(1)}\dot{x}^2\dot{y}+g_y^{(1)}\dot{x}\dot{y}^2+g_x^{(2)}\dot{x}\dot{y}^2 + g_y^{(2)}\dot{y}^3+2g^{(0)}\ddot{x}\dot{x}+g^{(1)}\ddot{y}\dot{x}+g^{(1)}\dot{y}\ddot{x}+2g^{(2)}\ddot{y}\dot{y} + h_x\dot{x}+h_y\dot{y}. \tag{4}$$

Regrouping and equating to zero the coefficients of each monomial in the velocities, we obtain at order three

$$g_x^{(0)}=0,\quad g_y^{(0)}+g_x^{(1)}=0,$$
$$g_y^{(1)}+g_x^{(2)}=0,\quad g_y^{(2)}=0. \tag{5}$$

The solution of this system of equations is straightforward:

$$g^{(0)}=\alpha y^2+\beta y+\gamma,$$
$$g^{(1)}=-2\alpha xy-\beta x-\delta y-\epsilon, \tag{6}$$
$$g^{(2)}=\alpha x^2+\delta x+\zeta.$$

At first order we obtain

$$h_x+2g^{(0)}\ddot{x}+g^{(1)}\ddot{y}=0,$$
$$h_y+g^{(1)}\ddot{x}+2g^{(2)}\ddot{y}=0. \tag{7}$$

The integrability condition for h reads thus:

$$\frac{\partial}{\partial\dot{y}}[2g^{(0)}V_x+g^{(1)}V_y]=\frac{\partial}{\partial x}[g^{(1)}V_x+2g^{(2)}V_y], \tag{8}$$

or, equivalently,

$$2(g^{(0)}-g^{(2)})V_{xy}+(2g_y^{(0)}-g_x^{(1)})V_x - (2g_x^{(2)}-g_y^{(1)})V_y-g^{(1)}(V_{xx}-V_{yy})=0. \tag{9}$$

This is the equation found by Darboux,[1] which must be obeyed by the potential for the system to be integrable.

The case analyzed by Darboux is obtained by using the general solution (6) for the $g^{(i)}$s, with the assumption $\alpha\neq 0$. In this case one can, through the adequate translations in the x and y directions, eliminate all the linear terms in (6). Furthermore, we observe that γ and ζ in $g^{(0)}$ and $g^{(2)}$ appear only through $\gamma-\zeta$ in (9). Thus one can take $\zeta=0$ without loss of generality. This is by no means unimportant because $\zeta=0$ ensures that no term of the form $\eta\dot{y}^2$ with η constant will be present in C. This means that C cannot be just a multiple of the total energy unless it vanishes identically. Finally, a rotation of coordinates can be performed to allow the elimination of ϵ unless $\epsilon^2+\gamma^2=0$, a case that Darboux also ignored. In the general case, Eq. (9) becomes with $\alpha=1$

$$xy(V_{xx}-V_{yy})+(y^2-x^2+\gamma)V_{xy}+3yV_x-3xV_y=0. \tag{10}$$

The solution of Eq. (10) was obtained by Darboux[1] as

$$V=f(u)-g(v)/(u^2-v^2), \tag{11}$$

where u and v are given by

$$2u^2=\rho^2+\gamma+[(\rho^2+\gamma)^2-4\gamma x^2]^{1/2},$$
$$2v^2=\rho^2+\gamma-[(\rho^2+\gamma)^2-4\gamma x^2]^{1/2},$$
$$\rho^2=x^2+y^2,$$

and f and g are two arbitrary functions.

In the degenerate case $\gamma=0$, the singular limit of the solution is

$$V=G(\rho)+\Phi(y/x)/\rho^2, \tag{12}$$

where G and Φ are also two arbitrary functions. If $\epsilon^2+\gamma^2=0$, Eq. (9) becomes

$$(xy\pm i\gamma/2)(V_{xx}-V_{yy}) + (y^2-x^2+\gamma)V_{xy}+3yV_x-3xV_y=0,$$

the general solution of which is

$$V=\{F[\rho^2+(\rho^4-2\gamma z^2)^{1/2}] + G[\rho^2-(\rho^4-2\gamma z^2)^{1/2}]\}/(\rho^4-2\gamma z)^{1/2}, \tag{13}$$

with $z=x\pm iy$, and where F and G are again two arbitrary functions. In the degenerate case $\gamma=0$, we recover the singular limit (12).

The case *not* analyzed by Darboux corresponds to $\alpha=0$. Before proceeding to examine this case in detail, let us deal with the particular case where, in addition, β and δ also vanish. One obtains from (9), with $\gamma=1$,

$$V_{xy}+\epsilon(V_{xx}-V_{yy})=0.$$

If $4\epsilon^2+1\neq 0$, the solution of this equation is straightforward. An adequate rotation reduces it to

$$V_{xy}=0,$$

which indicates a separable potential

$$V=f(x)+g(y), \tag{14}$$

where f and g are two arbitrary functions. If $\epsilon=\pm i/2$, this rotation becomes singular, the potential is not separable, and the general solution is

$$V=(x^2+y^2)F(x\pm iy)+G(x\pm iy), \tag{15}$$

where F and G are again two arbitrary functions. This potential will be called quasi separable.

We now turn to the case $\alpha=0$ but β (or δ) nonvanishing. Translations of x and y allow us to eliminate $\gamma-\zeta$ and ϵ. Putting $\lambda=-\delta/\beta$, we obtain

$$2yV_{xy}+3V_x+x(V_{xx}-V_{yy}) + \lambda[2xV_{xy}+3V_y-y(V_{xx}-V_{yy})]=0. \tag{16}$$

Finally an adequate rotation of coordinates allows the choice $\lambda=0$, which eliminates $\dot{x}^2$ in the constant of motion, unless $\lambda^2+1=0$, where this rotation becomes singular. (The choice $\lambda=\infty$ would correspond to eliminating $\dot{y}^2$ in the constant of motion). The case $\lambda^2+1=0$ will be treated in Sec. IV. In the general case, we obtain

$$2yV_{xy}+3V_x+x(V_{xx}-V_{yy})=0. \tag{17}$$

Several solutions to this PDE can be found by inspection, and the general solution will be presented in Sec. IV. One important class of solutions of Eq. (17) are the homogeneous polynomial solutions. They can be derived in an elementary way and turn out to be

$$V_0=1,\quad V_1=2y,\quad V_2=4y^2+x^2,$$
$$V_3=8y^3+4x^2y,\quad V_4=16y^4+12x^2y^2+x^4, \tag{18}$$
$$V_5=32y^5+32x^2y^3+6x^4y,\quad \text{etc.}$$

As we explained in Ref. 6, the mere existence of the quintic polynomial compelled us to weaken the Painlevé property criterion of integrability.

III. PAINLEVÉ ANALYSIS OF EQUATIONS OF MOTION

Let us consider the equations of motion associated with the integrable polynomial potential $-V_5/2^5$ [cf. Eq. (18)]:

$$\ddot{x} = 2y^3x + \tfrac{3}{2}yx^3,$$
$$\ddot{y} = 5y^4 + 3y^2x^2 + \tfrac{3}{16}x^4. \tag{19}$$

A leading order analysis in the complex t plane shows that a possible behavior of the solution near a singularity is

$$x,y \propto (t - t_0)^{-2/3}.$$

This, in itself, is *not* yet incompatible with the original conjecture as x^3 and y^3 could have been pure poles. However, the study of the resonances[7] shows that this is not the case. There is one resonance, namely $\frac{10}{3}$, which is positive and not an integer. This shows that $(t - t_0)^{2/3}x$ has in general an algebraic singularity. However, the expansion of x and y does not contain anything "worse" than powers of $(t - t_0)^{1/3}$, which is *still* quite remarkable.

The fact that the occurrence of such "natural" algebraic singularities does not compromise integrability suggests a generalization of the ARS conjecture[7] for two-dimensional systems. We will say that a system has the "weak-Painlevé property" whenever the solution, in the neighborhood of a singularity at t_0, can be expressed as an expansion in powers of $(t - t_0)^{1/r}$, where r is an integer to be defined below. The new conjecture is now that two-dimensional integrable systems possess the weak-Painlevé property.

In order to define the exponent $1/r$, let us consider a polynomial potential of degree $p + 2$. The Painlevé analysis proceeds in two steps. First, one studies the homogeneous, highest order, part of the potential. Whenever the Painlevé property is satisfied, one looks for lower order terms that do not destroy it. Incidentally, the same two steps procedure is used when one attempts a direct calculation of the integrals of motion.

Thus, to begin with, we restrict ourselves to a homogeneous polynomial potential of degree $p + 2$. It is obvious that the leading behavior in the neighborhood of a singularity will be

$$x,y \propto (t - t_0)^{-2/p}.$$

Moreover, there is *always* a resonance at $2 + 4/p$, which, except for $p = 1$, 2, and 4, is *not* an integer. Thus, there is no hope for $(t - t_0)^{2/p}x$ to be regular. In general, the solution will be an expansion in powers of $(t - t_0)^{1/p}$, which shows that one can take $r = p$. Indeed, further studies[10] have revealed a need to refine the definition of the "natural" power r. In this paper, however, the choice $r = p$ is sufficient, as our aim here is to study a specific family of "weak-Painlevé" potentials. In the following paper,[10] we will deal with the exhaustive search of weak-Painlevé integrable polynomials of degree 3. Since the investigation is more and more cumbersome with increasing order, we do not try to be exhaustive in the present work, but to show how useful the heuristic power of the weak-Painlevé property is.

Let V be a homogeneous polynomial of degree $p + 2$. We do not lose much generality by assuming that through a suitable rotation, the coefficient of xy^{p+1} can be set to zero while that of y^{p+2} does not vanish. We write V as

$$V = -\sum_{k=0}^{p+2} a_k x^k y^{p+2-k},$$

with $a_1 = 0$. The equations of motion are

$$\ddot{y} = -V_y = \sum_{k=0}^{p+1} (p + 2 - k)a_k x^k y^{p+1-k}, \tag{20}$$

$$\ddot{x} = -V_x = \sum_{k=2}^{p+2} k a_k x^{k-1} y^{p+2-k}. \tag{21}$$

In general, there are singularities in the neighborhood of which both x and y behave as $(t - t_0)^{-2/p}$. In addition, there are singularities near which y still behaves as $(t - t_0)^{-2/p}$, while x need not diverge that fast. Indeed, since a_1 vanishes, the right-hand side of Eq. (21) need not diverge as $(t - t_0)^{-2-2/p}$ but only as $x(t - t_0)^{-2}$, whatever the behavior of x is, provided it is *less* singular than that of y. Whenever it happens that $a_{p+1} = 0$ while $a_{p+2} \neq 0$, there will also be a singularity where x behaves as $(t - t_0)^{-2/p}$ while y does not diverge that fast. We must keep this possibility in mind, but we will *not* treat x and y on the same footing as the rotation has been explicitly chosen in order to have $a_1 = 0$.

Let us first consider the singularity in the neighborhood of which y diverges as $(t - t_0)^{-2/p}$ but not x. Then Eq. (20) fixes the coefficient of the leading term of y. We write

$$y = b(t - t_0)^{-2/p} + \epsilon(t - t_0)^{-2/p},$$

where $\epsilon \to 0$ as $t \to t_0$, and by equating the fastest diverging terms we find

$$2(p + 2)b/p^2 = (p + 2)a_0 b^{p+1}.$$

Since y must actually diverge as $(t - t_0)^{-2/p}$, $b \neq 0$ and thus

$$a_0 b^p = 2/p^2.$$

The possible behaviors for x are determined by Eq. (21). Let γ be the power dependence of x, i.e.,

$$x \propto (t - t_0)^{\gamma},$$

we find by equating the fastest diverging terms

$$\gamma(\gamma - 1) = 2a_2 b^p.$$

A first necessary condition for the weak-Painlevé property to be satisfied is that $\gamma = u/p$ with u an integer (positive or negative). This gives the equation

$$4a_2/a_0 = u(u - p). \tag{22}$$

One must now compute the resonances.[7] This is rather easy since the problem separates into resonances for y, which are -1 and $2 + 4/p$, and resonance for x which are 0 and $1 - 2u/p$. We do not get any further conditions since we already need u to be an integer. The only powers of $(t - t_0)$ that will appear at the resonances are multiples of $1/p$. We have not yet shown that no logarithms will get into the picture, but we are free of "nonnatural" powers of $(t - t_0)$, for this type of singularity, at least.

We note that -1 corresponds to the arbitrariness of t_0

and 0 to the arbitrariness of the coefficient of $(t-t_0)^{u'/p}$ in x. The equation for a resonance n of y is only in terms of the quantity $N=(n-2/p)(n-1-2/p)$. Since $n=-1$ is always a resonance, it follows that $N_1=(1+2p)(2+2/p)$ satisfies the resonance condition; hence $2+4/p$ is also always a resonance as we noted above. On the other hand, the possible behaviors for x are $(t-t_0)^{u'/p}$, where u satisfies Eq. (22). There are in general two roots to this equation, related by

$$u'+u''=p.$$

Note that at least one of these roots is positive, thus strictly larger than -2, and the corresponding behavior of x is less divergent than that of y. The fact that if x behaves as $(t-t_0)^{u'/p}$, the resonance is $1-2u'/p$ expresses that

$$u''/p=1-u'/p=u'/p+(1-2u'/p),$$

i.e., this resonance corresponds to the freedom of the coefficient of $(t-t_0)^{u''/p}$ in x. In fact, this resonance is only actually present if

$$u''>u'>-2;$$

otherwise, it is purely formal.

Let us now turn to the case where both x and y diverge as $(t-t_0)^{-2/p}$, i.e.,

$$x=\alpha(t-t_0)^{-2/p}+\epsilon_1(t-t_0)^{-2/p},$$
$$y=\beta(t-t_0)^{-2/p}+\epsilon_2(t-t_0)^{-2/p},$$

where ϵ_1 and ϵ_2 go to zero as $t\to t_0$. We choose the following notation: Whenever V or one of its partial derivatives has an overbar, it means that it is estimated at $x=\alpha$, $y=\beta$. Then we get the following equations for α and β, by equating the most diverging terms in Eq. (20) and (21):

$$2(2+p)\alpha/p^2=-\overline{V}_x,$$
$$2(2+p)\beta/p^2=-\overline{V}_y. \tag{23}$$

In general, this system can have several couples (α,β) as solutions. For a given choice of (α,β) that satisfies Eqs. (23), a resonance n will occur whenever the determinant of the matrix M vanishes, where M is given by

$$M=\begin{pmatrix}(n-2/p)(n-1-2/p)+\overline{V}_{xx} & \overline{V}_{xy}\\ \overline{V}_{xy} & (n-2/p)(n-1-2/p)+\overline{V}_{yy}\end{pmatrix}.$$

Note that n only enters through the expression $N=(n-2/p)(n-1-2/p)$. Again, $n=-1$ is always a resonance; thus $N_1=(1+2/p)(2+2/p)$ must be a solution, as can be checked directly. Indeed, since V_x is a homogeneous polynomial of degree $p+1$,

$$xV_{xx}+yV_{xy}=(p+1)V_x.$$

Thus, for any couple (α,β) that satisfies Eqs. (23),

$$\alpha\overline{V}_{xx}+\beta\overline{V}_{xy}=-2(p+2)(p+1)\alpha/p^2=N_1\alpha,$$

and similarly

$$\alpha\overline{V}_{xy}+\beta\overline{V}_{yy}=-2(p+2)(p+1)\beta/p^2=-N_1\beta.$$

With the choice $N=N_1$, the matrix M becomes

$$M=\begin{pmatrix}-(\alpha/\beta)\overline{V}_{xy} & \overline{V}_{xy}\\ \overline{V}_{xy} & -(\beta/\alpha)\overline{V}_{xy}\end{pmatrix},$$

and its determinant indeed vanishes. This shows that $2+4/p$ is also always a resonance as it leads to the same value N_1 for N. The other value N_2 of N for which the determinant vanishes obviously satisfies

$$N_2+N_1=-\overline{V}_{xx}-\overline{V}_{yy}.$$

A second necessary condition for the weak-Painlevé property to be satisfied is that N_2 leads to resonances n that are of the form s/p, with s an integer. This can be written

$$-p^2(\overline{V}_{xx}+\overline{V}_{yy})=(s-2)(s-2-p)+(2+p)(2+2p). \tag{24}$$

This equation must be satisfied for some integer s for each choice of a couple (α,β) that solves Eqs. (23). The integer s could very well depend on the choice of the couple (α,β). It is in general difficult to find *all* the solutions to this problem, even for small values of p. But one can find *some* solutions by using a simplifying assumption. Since $a_1=0$, V_x has no term proportional to y^{p+1}, and x thus can be factored out of V_x. Then V_x/x is a homogeneous polynomial in x and y of degree p, which is also true of $V_{xx}+V_{yy}$. The simplifying assumption we will make is that these polynomials are proportional, their ratio being chosen in such a way as to satisfy Eq. (24). This is a very strong restriction. It implies that the resonances are the same for all choices of the couple (α,β), and this is certainly not a necessary condition. Still, it will prove very rich though not exhaustive. If we write

$$V_{xx}+V_{yy}=\lambda V_x/x, \tag{25}$$

Eq. (24) becomes

$$-\lambda p^2\overline{V}_x/\alpha=(s-2)(s-2-p)+(2+2p),$$

and since α and β satisfy Eqs. (23)

$$2\lambda(2+p)=(s-2)(s-2-p)+(2+p)(2+2p). \tag{26}$$

Equation (25) leads to a recursion relation between a_{k+2} and a_k. This recursion relation is

$$(k+2)(k+1)a_{k+2}+(p+2-k)(p+1-k)a_k=\lambda(k+2)a_{k+2}. \tag{27}$$

If λ is a positive odd integer strictly smaller than $p+2$, there are no solutions with $a_0\neq 0$, because the coefficient of $a_{\lambda+1}$ will vanish. Such values of λ must be discarded. In general, only a's with even indices do not vanish, unless λ is a nonzero even integer strictly smaller than $p+2$, in which case $a_{\lambda+1}$ need not vanish, although $a_{\lambda-1}=0$. Thus the polynomial V is entirely determined by λ, up to the multiplicative arbitrary constant a_0 or, in the special case above, by two arbitrary constants a_0 and $a_{\lambda+1}$.

From Eq. (27), the ratio a_2/a_0 is given by

$$a_2/a_0=(p+2)(p+1)/2(\lambda-1).$$

Comparing with Eq. (22), we see that λ must satisfy

$$\lambda=1+2(p+2)(p+1)/[u(u-p)], \tag{28}$$

with u an integer. The problem is now to find two integers u

and s that satisfy simultaneously Eqs. (26) and (28).

If p is odd, a_{p+1} does not vanish, and there is no singularity where x diverges faster than y. If p is even, on the other hand, a_{p+1} in general vanishes, and there will be such a singularity, unless

$$2 \leqslant \lambda \text{ (even)} < p + 2. \tag{29}$$

A necessary condition for this singularity to be of the weak-Painlevé type is

$$4a_p/a_{p+2} = v(v-p),$$

with v an integer, in analogy to Eq. (22). From Eq. (27), this can be written

$$v(v-p) = 2(p+2)(\lambda - p - 1). \tag{30}$$

If p is even, one must thus solve (26), (28), and (29) or (30), while (26) and (28) suffice for p odd.

For every p there are always at least two solutions to this problem [note that solutions depend only on $(s-2)$ $(s-2-p)$ and $u(u-p)$, $v(v-p)$]:

(i) $u = p+2$ (or -2), $s = 0$ (or $p+4$),

with $\lambda = p+2$,

(ii) $u = p+1$ (or -1), $s = 2p+6$ (or $-p-2$),

with $\lambda = 2p+5$,

as can easily be checked. These solutions, for even p, do lead to integer values of v, $p+2$ (or -2), and $2p+4$ (or $-p-4$), respectively.

There might be other solutions for some values of p, but none were found for small values of p except $p = 1$, $u = 4$ (or -3), $s = 2$ (or 3), with $\lambda = 2$. This leads to the following potential:

$$V = a_0(y^3 + 3yx^2) + a_3x^3,$$

which is known to be integrable (separable).

Let us now consider the first solution. In that case there is a resonance $n = s/p$ equal to 0. This should mean that one of the coefficients α and β is arbitrary. Now, for even p, this is true. Substituting $\lambda = p+2$ into Eq. (27) allows us to compute V, and one finds

$$V = a_0(y^2 + x^2)^{p/2}.$$

Thus only $(\alpha^2 + \beta^2)$ is determined and the occurrence of the resonance $n = 0$ is normal. This potential is obviously integrable. This case, however, is not as trivial as it seems, because one can now add a lower degree polynomial that need *not* be rotationally invariant, without destroying integrability. One thus recovers exactly the polynomial potentials of the family found by Darboux,[1] which are indeed all even. For p odd, on the other hand, the resonance $n = 0$ is pathological, as it is not related to the arbitrariness of either α or β. If one solves Eqs. (20) and (21), one finds

$$\alpha = 0, \quad \beta = b.$$

This is not at all acceptable, as α may not vanish, since we assumed that x does behave as $(t - t_0)^{-2/p}$. Although we found a "formal" solution to the arithmetical equations, we do not have in this case a weak-Painlevé potential. In fact, the vanishing of α entails here a logarithmic singularity. For $p = 1$, this "pseudo-Painlevé" case has been noted before.[11]

The new and interesting case is

$$u = -1 \text{ (or } p+1), \quad n = 2p+6, \text{ (or } -p-2),$$
$$\text{with } \lambda = 2p+5,$$

complemented by $v = 2p+4$ (or $-p-4$) for even p. One can solve the recursion relation Eq. (27) with $a_0 = 2^{p+2}$ to find

$$a_{2k} = 2^{p+2-2k} c_{p-k}^{k},$$

for $0 \leqslant 2k \leqslant p+2$, $a_{2k+1} = 0$.

This family of polynomials are indeed integrable. Note that, here, these homogeneous potentials are already nontrivial. They coincide with the polynomials of Eq. (18). To see that, we rewrite Eq. (25) as

$$x(V_{xx} + V_{yy}) = (2p+5)V_x,$$

and note that V_x is homogeneous of degree $(p+1)$, i.e.,

$$(p+1)V_x = xV_{xx} + V_{xy};$$

thus one recovers Eq. (17),

$$x(V_{yy} - V_{xx}) - 2yV_{xy} = 3V_x.$$

The homogeneous polynomials we have found satisfy a set of necessary conditions for the weak-Painlevé property to be true. To check that they are sufficient, one should ascertain that no logarithms appear at the resonances. This had indeed been found to hold for some low values of p. It would be straightforward, although increasingly tedious, to verify this property for higher values of p. It will be even more cumbersome, except for very small values of p, to check that the only lower degree terms one can add without destroying the weak-Painlevé properly are the polynomials that solve the same Eq. (17). Actually these two properties (absence of logarithms and superposition of solutions) are expected to hold for all p, because Eq. (17) is just the integrability condition. There is no need to prove that rigourously. As a matter of fact, the ARS conjecture is meant to be a heuristic tool rather than a rigorous approach. In the case at hand it has well served its purpose by leading to the same integrability condition (17) as the direct approach.

IV. SOLUTION OF THE PDE TO BE SATISFIED BY THE POTENTIAL

The solutions of Eq. (17) can be obtained in a variety of ways (Fourier transform, use of characteristics, etc.). However, given the particular form of the equation and the intuition stemming from the knowledge of particular solutions, a direct approach appears simpler. We start by introducing a new set of variables

$$\rho = (x^2 + y^2)^{1/2}, \quad \eta = y. \tag{31}$$

A straightforward calculation obtains

$$V_{\rho\rho} + 2V_\rho/\rho - V_{\eta\eta} = 0. \tag{32}$$

We do away with the first-order derivative by introducing

$$U = \rho V.$$

There results the equation

$$U_{\rho\rho} - U_{\eta\eta} = 0, \tag{33}$$

which is the well known form of the wave equation. Its solu-

tion reads

$$U = f(\rho + \eta) + g(\rho - \eta),$$

with f and g arbitrary functions. So the solution for the potential V becomes

$$V = [f(\rho + \eta) + g(\rho - \eta)]/\rho. \tag{34}$$

One can easily check from this form that this family of integrable potentials cannot be contained in the Darboux solution (11), even though the intersection of the two sets is not empty. Once the form of the potential is obtained, we can integrate Eqs. (7) and deduce the form of the integral of motion. We start by rewriting Eqs. (7) with respect to the variables ρ and η:

$$h_\rho = \eta V_\rho - \rho V_\eta, \quad h_\eta = \eta V_\eta - \rho V_\rho.$$

The integration is straightforward once one expresses V through Eq. (34), and we thus find

$$h = [(\rho + \eta)g(\rho - \eta) - (\rho - \eta)f(\rho + \eta)]/\rho. \tag{35}$$

The same results Eqs. (34) and (35) would have been obtained in a more systematic (and tedious) way by looking for the characteristics of Eq. (17), which in view of Eqs. (31), define a system of parabolic coordinates.

In the special case $\lambda^2 + 1 = 0$, Eq. (16) becomes

$$2\bar{z}V_{\bar{z}\bar{z}} + 3V_{\bar{z}} = 0,$$

with $z = x \pm iy$, $\bar{z} = x \mp iy$. The general solution is

$$V = F(x \pm iy)/(x^2 + y^2)^{1/2} + G(x \pm iy), \tag{36}$$

where F and G are two arbitrary functions.

With Eqs. (11)–(15), the results (34) and (36) complete the answer to the question of the existence of an integral of motion quadratic in velocities for the motion of a particle in two dimensions. It is, however, instructive to examine some particular families of solutions.

(a) Let us start with the potential

$$V = 1/\rho.$$

This potential is manifestly contained in Eq. (34), and at the same time, from Eq. (12), belongs to the Darboux family for $\gamma = 0$. Its Darboux integral is simply

$$C_1 = (x\dot{y} - y\dot{x})^2,$$

i.e., the square of the angular momentum. However from Eq. (35) another integral of motion can be computed

$$C_2 = \dot{y}\dot{x}^2 - x\dot{x}\dot{y} + y/\rho$$

or

$$C_2 = \dot{x}(x\dot{y} - y\dot{x}) + y/(x^2 + y^2)^{1/2}.$$

This just the y component of the quantity

$$\dot{\mathbf{r}} \times \mathbf{L} + \mathbf{r}/|\mathbf{r}|,$$

which is the third integral of motion in the case of the Kepler problem. The existence of an additional integral in this case is associated with the degeneracy of the problem, which is manifested, for instance, by the separability in more than one system of coordinates. From a physical point of view, the usefulness of this invariant resides in the fact that it helps fix the orientation of the orbit in the plane.

(b) Polynomial solutions: In Ref. 6, we already presented the homogeneous polynomial solutions of Eq. (17). Their general form is

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} C^k_{n-k} x^{2k} y^{n-2k}, \tag{37}$$

where

$$C^k_{n-k} = \binom{n-k}{k} = (n-k)!/k!(n-2k)!.$$

We can obtain them directly from Eq. (34) by choosing

$$f(q) = (-1)^n g(q) = q^{n+1}/2. \tag{38}$$

The general solution can be built from a superposition of the homogeneous polynomials of each degree, due to the linearity of Eq. (17). The simplest polynomial solution to Eq. (17) (apart from $V_0 = 1$) is, of course, $V_1/2 = y$. This means that a multiple of y can always be added to the potential. In the case of polynomial potentials this fact is quite useful as it allows us the freedom of a translation in the y direction without the appearance of a linear term in the potential. A polynomial potential case of particular interest is the Hénon–Heiles potential for which Greene[4] obtained the second integral of motion. In this case of integrability, the Hénon–Heiles potential reads

$$V = (Ax^2 + By^2)/2 + x^2 y + 2y^3.$$

It can be shown that this potential reduces to a superposition of the potentials V_3, V_2, and V_1 of Eq. (37) after the adequate translation.

(c) Homogeneous solutions: If we allow negative values of n in Eq. (38) one finds rational integrable potentials, which can be related to the polynomials V_n by

$$V_{-n-2} = (-1)^n V_n/x^{(2n+2)}.$$

In the case $n = -1$ one would have $f = -g = 1/2$ and V_{-1} vanishes. In addition to this family of homogeneous rational potentials there exists a second homogeneous family W_n given by $f(q) = (-1)^{n+1} g(q) = q^{n+1}/2$, which can be related between themselves by

$$W_{-n-2} = (-1)^{n+1} W_n/x^{(2n+2)}.$$

For $n \geq 0$, ρW_n is an homogeneous polynomial in x, y of degree $n + 1$, and W_{-1} is just $1/\rho$.

We thus find two potentials homogeneous of order n for all n, except $n = -1$ where we have as yet only one since $V_{-1} = 0$. Actually, a second potential homogeneous of order -1 exists and is

$$Z = \ln[(\rho + y)/(\rho - y)]/\rho,$$

or

$$Z = 2 \ln \tan(\tfrac{1}{2} \arctan y/x + \pi/4)/\rho.$$

This exhausts all homogeneous solutions of integer ($\gtrless 0$) order.

V. CONCLUSION AND OUTLOOK

The main result of this paper is the identification of a whole new family of integrable dynamical systems for one particle in two space dimensions. This was achieved through the derivation and resolution of a PDE to be satisfied by the potential in order to ensure the existence of an integral of motion quadratic in the velocities.

The most important point is that this equation was derived in two different ways. The first was a direct calculation based on the search for a constant of the motion. The second was through a suitable implementation of the "weak-Painlevé" property of the equations of motion, which we introduced in a previous publication.[6] This constitutes a first evidence of the usefulness of the weak-Painlevé concept.

In a companion paper[10] we will have the occasion to deal in more detail with this notion. We will actually show that the weak-Painlevé property is a very efficient tool for the identification of integrable cases. More precisely, we were able to find *all* weak-Painlevé case for third degree polynomial potentials, the integrability being confirmed in each case by direct computation of the integral of motion. In the case of fourth-degree polynomials, new cases of integrability will be presented. Finally we will focus on the precise definition of the weak-Painlevé property, and we will insist on its conditions of applicability and the unavoidable "caveat."

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Integrability of Hamiltonians with third- and fourth-degree polynomial potentials[a)]

B. Grammaticos and B. Dorizzi
Département de Mathématiques (M.T.I.), Centre National d'Etudes de Télécommunications, 92131 Issy les Moulineaux, France

A. Ramani
C. P. Th., Ecole Polytechnique, 91128 Palaiseau Cedex, France

(Received 12 October 1982; accepted for publication 17 December 1982)

The weak-Painlevé property, as a criterion of integrability, is applied to the case of simple Hamiltonians describing the motion of a particle in two-dimensional polynomial potentials of degree three and four. This allows a complete identification of all the integrable cases of cubic potentials. In the case of quartic potentials, although our results are not exhaustive, some new integrable cases are discovered. In both cases the integrability is explicited by a direct calculation of the second integral of motion of the system.

PACS numbers: 02.30.Jr

I. INTRODUCTION

In this paper we carry on our investigations on the integrability of dynamical systems based on the generalization of the Painlevé criterion we introduced in Ref. 1. In systems with more than one degree of freedom integrability is of rare occurrence. (Integrability, in the case of Hamiltonian systems, is defined as the existence of analytical, single-valued integrals of motion, in number equal to that of the degrees of freedom of the system, time independent and in involution.) The detection of integrability has been greatly facilitated these last years by the work of Ablowitz, Ramani, and Segur.[2] They have conjectured, and verified in a multitude of cases of physical interest, that integrability is intimately related to the analytic properties of the solutions of the equations of motion. Namely, whenever the solutions possess the Painlevé property, i.e., their only movable singularities on the complex time plane are poles, the system is integrable. The reciprocal is also true, the Painlevé property having been verified for the known integrable systems (eventually after some, trivial or not, changes of variables). However, although the ARS conjecture holds true for many (or infinite) degrees of freedom systems, it became clear from the results presented in Ref. 1 that some revision of the Painlevé criterion was needed for two-dimensional systems. (Actually, for one-dimensional systems, the Painlevé criterion is superfluous: They are integrable by definition). We discovered, namely, a quintic polynomial Hamiltonian in two dimensions, which did not possess the Painlevé property in the classical manner while being integrable. The integrability was ensured by the explicit calculation of the second integral of motion. However, some regularity still persisted: The solution in the neighborhood of a singularity could be expressed as an expansion in powers of $(t - t_0)^{1/3}$. This led us to the introduction of the "weak Painlevé" notion as a criterion of integrability. The "weak Painlevé property" was associated to an expansion in powers of $(t - t_0)^{1/r}$, with r being a "natural" power, determined solely by the dominant terms in the equations of motion.

In the preceding paper, Ref. 3, we presented a first application of the weak Painlevé concept. A partial differential equation, obeyed by a certain class of integrable potentials, was derived, based on the weak Painlevé property, which we ensured in the easiest way through some simplifying assumptions. The same equation was derived through a direct search for integrals of motion quadratic in velocities, and its solution provided a whole new class of integrable systems, to which the quintic polynomial, mentioned above, belongs.

This twofold approach, integrability prognostication based on the Painlevé criterion, and integrability verification through a direct calculation of the constants of motion is pursued in the present work as well. By deleting some of the simplifying assumptions we had adopted in Ref. 3, we are able to extend our study of integrability to a wider class of potentials. However, as the calculations become very rapidly prohibitively intricate, one is compelled to restrict oneself to some low-order polynomial potentials. The aim of the present paper is to present our results on third and fourth degree polynomial potentials. The direct method for the search of integrals of motion is first introduced in Sec. II. Sections III and IV are devoted respectively to the study of third and fourth degree Hamiltonians. Section V finally deals with the definition of the "natural" power we alluded to previously, definition which is the cornerstone of the "weak Painlevé" property.

II. DIRECT SEARCH FOR INTEGRALS OF MOTION

In Ref. 3 we presented a general method for the investigation of the existence of integrals of motion quadratic in the velocities, following Bertrand's approach.[4] In the present work we are going to focus on integrals quartic in the velocities. For the sake of completeness we start by sketching briefly the case of a constant of motion cubic in velocities.

We start with a form:

$$C = f_0\dot{x}^3 + f_1\dot{x}^2\dot{y} + f_2\dot{x}\dot{y}^2 + f_3\dot{y}^3 + g_0\dot{x} + g_1\dot{y}. \qquad (2.1)$$

a) The authors dedicate this work to their Soviet colleague, the mathematician and physicist Salomon Alber.

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No even power in the velocities terms are allowed, as we suppose that the Hamiltonian is invariant under time reversal. The condition $dC/dt = 0$ leads to a system of partial differential equations obtained by equating to zero the coefficients of each monomial $\dot{x}^m\dot{y}^n$. We thus obtain

$$\frac{\partial f_0}{\partial x} = 0, \quad \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial x} = 0, \quad \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0,$$
$$\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x} = 0, \quad \frac{\partial f_3}{\partial y} = 0. \tag{2.2}$$

The solution to Eq. (2.2) is straightforward. The f_i's are cubic polynomials in (x,y):

$$\begin{aligned} f_0 &= \alpha y^3 + \beta y^2 + \gamma y + \delta, \\ f_1 &= -(3\alpha y^2 + 2\beta y + \gamma)x + \epsilon y^2 + \zeta y + \eta, \\ f_2 &= (3\alpha y + \beta)x^2 - (2\epsilon y + \zeta)x + \theta y + \kappa, \\ f_3 &= -\alpha x^3 + \epsilon x^2 - \theta x + \lambda. \end{aligned} \tag{2.3}$$

The next set of equations reads

$$3f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0,$$
$$2f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0, \tag{2.4}$$
$$f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} = 0.$$

One replaces f_i from (2.3) and $\ddot{x},\ddot{y}$ from the equations of motion, $\ddot{x} = -\partial V/\partial x$, $\ddot{y} = -\partial V/\partial y$, and integrates for g_i. The compatibility condition for the integration to be possible reads

$$\frac{\partial^2}{\partial x^2}(f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^2}{\partial x\partial y}(2f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^2}{\partial y^2}(3f_0\ddot{x} + f_1\ddot{y}) = 0. \tag{2.5}$$

Once the g_i are obtained there remains a last equation to be verified:

$$g_0\ddot{x} + g_1\ddot{y} = 0, \tag{2.6}$$

which, in view of (2.4), is a nonlinear PDE for the potential V. So, for the system to possess an integral of motion cubic in velocities, the potential must satisfy Eqs. (2.5) and (2.6). This actually can be realized in some cases, as, for example, the Toda system,[5,6] and will be the object of a future work. However, none of the systems examined in this paper fall into this class and we are thus led to consider integrals of motion quartic in velocities.

Let us look for an integral of motion of the form:

$$C = f_0\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 + g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h. \tag{2.7}$$

Again the first set of partial differential equations for f_i can be solved in a straightforward way giving:

$$\begin{aligned} f_0 &= \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \epsilon, \\ f_1 &= -(4\alpha y^3 + 3\beta y^2 + 2\gamma y + \delta)x + \zeta y^3 + \eta y^2 + \theta y + \rho, \\ f_2 &= (6\alpha y^2 + 3\beta y + \gamma)x^2 - (3\zeta y^2 + 2\eta y + \theta)x + \kappa y^2 + \lambda y + \mu, \\ f_3 &= -(4\alpha y + \beta)x^3 + (3\zeta y + \eta)x^2 - (2\kappa y + \lambda)x + \nu y + \xi, \\ f_4 &= \alpha x^4 - \zeta x^3 + \kappa x^2 - \nu x + \sigma. \end{aligned} \tag{2.8}$$

As we have explained in Ref. 3, the Hamiltonian, being a constant of motion, can be freely added to C. The same applies to the square of the Hamiltonian as well, which allows us one of the three choices $\epsilon = 0$ or $\mu = 0$ or $\sigma = 0$.

The PDE's for g_i read

$$4f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0,$$
$$3f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0,$$
$$2f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} = 0, \tag{2.9}$$
$$f_3\ddot{x} + 4f_4\ddot{y} + \frac{\partial g_2}{\partial y} = 0.$$

As in the case of cubic integrals, for Eqs. (2.9) to be integrable, the potential must satisfy a compatibility condition:

$$-\frac{\partial^3}{\partial x^3}(f_3\ddot{x} + 4f_4\ddot{y}) + \frac{\partial^3}{\partial x^2\partial y}(2f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^3}{\partial x\partial y^2}(3f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^3}{\partial y^3}(4f_0\ddot{x} + f_1\ddot{y}) = 0. \tag{2.10}$$

Next, the equations for h are obtained:

$$2g_0\ddot{x} + g_1\ddot{y} + \frac{\partial h}{\partial x} = 0,$$
$$g_1\ddot{x} + 2g_2\ddot{y} + \frac{\partial h}{\partial y} = 0. \tag{2.11}$$

The compatibility condition for this last system reads:

$$\frac{\partial}{\partial y}(2g_0\ddot{x} + g_1\ddot{y}) = \frac{\partial}{\partial x}(g_1\ddot{x} + 2g_2\ddot{y}). \tag{2.12}$$

So an integral of motion quartic in velocities exists whenever the potential satisfies the PDE's (2.10) and (2.12), the latter being a nonlinear one. The search is somewhat facilitated by the fact that in all nontrivial known cases of integrable potentials the f_i are just constants, independent of x and y. In that case equation (2.10) [and (2.5) as well] can be easily solved. As a matter of fact, for third degree polynomial potentials, Eq. (2.10) is automatically satisfied with constant f_i's while, for fourth degree potentials, Eq. (2.10) amounts to just one constraint.

III. THIRD DEGREE POLYNOMIAL POTENTIALS

As we have explained in the previous section, the integrability conditions for the motion of a particle in a two-dimensional potential take the form of a system of two PDE's, one of which is nonlinear. The search for the general solution of this system would have been an unrealistic attempt, given the complexity of the problem. One can look, however, at cases of physical interest with potentials of form simple enough for the problem to be tractable.

In this section we will examine in detail the case of a homogeneous polynomial potential of third degree. Our Hamiltonian can thus be viewed as a generalization of the Hénon–Heiles one.[7] The general form of the potential (within a global multiplicative factor which can be absorbed by a proper rescaling of time or space) is

$$V = y^3 + ay^2x + bx^2y + cx^3. \tag{3.1}$$

This form can be somewhat simplified through a rotation. It can be easily verified that one can almost always find a rotation which allows one to eliminate the y^2x term while preserving the y^3 one. There exist just two exceptions:

$$V = (x \pm iy)^3$$

and

$$V = (x^2 + y^2)(x \pm iy). \tag{3.2}$$

Incidentally, both potentials (3.2) are integrable. The first is manifestly separable while the second belongs to the quasiseparable case we introduced in Eq. (15) of Ref. 3. Both possess a second integral of motion quadratic in velocities. From now on, we limit ourselves to potentials of the form

$$V = y^3 + bx^2y + cx^3. \tag{3.3}$$

A. Painlevé analysis of the equations of motion

We will very closely follow the calculations of Ref. 3. The equations of motion are

$$\ddot{y} = -3y^2 - bx^2, \tag{3.4}$$

$$\ddot{x} = -2bxy - 3cx^2. \tag{3.5}$$

There are singularities in the neighborhood of which x and y both behave as $(t - t_0)^{-2}$. In addition, there are singularities where y behaves as $(t - t_0)^{-2}$ but x does not diverge that fast. Let us first consider the latter case. Equation (3.4) fixes the coefficient of the leading term of y. We write

$$y = \gamma(t - t_0)^{-2} + \epsilon(t - t_0)^{-2},$$

where $\epsilon \to 0$ as $t \to t_0$, and by equating the fastest diverging terms we find

$$6\gamma = -3\gamma^2.$$

Since y must actually diverge as $(t - t_0)^{-2}$, $\gamma \neq 0$, and thus

$$\gamma = -2.$$

The possible behaviors for x are determined by (3.5). Let s be the power dependence of x, i.e.

$$z \propto (t - t_0)^s;$$

we find, by equating the fastest diverging terms,

$$s(s - 1) = 4b. \tag{3.6}$$

We should demand that s be an integer because here weak Painlevé and Painlevé coincide. However, if $c = 0$, the equation for y contains only even powers of x, the equation for x contains only odd powers of x, and the system can be written in terms of y and x^2. In that case, the Painlevé property may be satisfied by x^2 rather than x, and this means that s may be a half-integer but *only* for $c = 0$. The search for resonances does not lead to any new condition (Ref. 3).

Let us now consider the case where both x and y diverge as $(t - t_0)^2$, i.e.,

$$x = \alpha(t - t_0)^{-2} + \epsilon_1(t - t_0)^{-2},$$

$$y = \beta(t - t_0)^{-2} + \epsilon_2(t - t_0)^{-2}.$$

where ϵ_1 and ϵ_2 go to zero as $t \to t_0$.

The equations for α and β are

$$6\beta = -3\beta^2 - b\alpha^2, \tag{3.7}$$

$$6\alpha = -2b\alpha\beta - 3c\alpha^2; \tag{3.8}$$

again α may not vanish as x actually diverges as $(t - t_0)^{-2}$. Thus (3.8) may be rewritten:

$$6 = -2b\beta - 3c\alpha. \tag{3.9}$$

In general, this system has two couples of solution (α, β). For each of them, a resonance n will occur whenever the determinant of the matrix M vanishes, where M is given by

$$M = \begin{pmatrix} (n-2)(n-3) + 2b\beta + 6c\alpha & +2b\alpha \\ +2b\alpha & (n-2)(n-3) + 6\beta \end{pmatrix}.$$

Note that n only enters through the expression $N = (n - 2)(n - 3)$. One solution is $N = 12$, which corresponds to $n = -1$ and $n = 6$. Indeed, using (3.7) and (3.9), the matrix M becomes

$$M = \begin{pmatrix} -2b\beta & 2b\alpha \\ 2b\alpha & -2b\alpha^2/\beta \end{pmatrix},$$

and its determinant clearly vanishes. The other solution N' for N satisfies:

$$N' + 12 = -(2b\beta + 6c\alpha + 6\beta).$$

Using once more Eq. (3.9), one finds

$$N' = (2b - 6)\beta. \tag{3.10}$$

The two choices of the couple (α, β) lead to two values of N'. Substituting α from Eq. (3.9) into (3.7), we find a second-order equation for β:

$$\beta^2(27c^2 + 4b^3) + \beta(54c^2 + 24b^2) + 36b = 0.$$

Its solutions β_1 and β_2 are such that

$$\beta_1 + \beta_2 = -\frac{54c^2 + 24b^2}{27c^2 + 4b^3} = -2 + \frac{(2b - 6)4b^2}{27c^2 + 4b^3},$$

$$\beta_1\beta_2 = \frac{36b}{27c^2 + 4b^3}.$$

The corresponding solutions N_1 and N_2 of (3.10) can be straightforwardly shown to satisfy

$$N_1 + N_2 = -2(2b - 6) + N_1N_2b/9. \tag{3.11}$$

This can be further symmetrized by remembering Eq. (3.6) and calling N_3 the quantity $s(s - 1)$. There results

$$36(N_1 + N_2 + N_3 - 12) = N_1N_2N_3. \tag{3.12}$$

A necessary condition for the Painlevé property to be satisfied is that N_1 and N_2 also be of the form $p_i(p_i - 1)$ with p_i integer and $i = 1$ or 2.

Remember that s should also be an integer except if $c = 0$, in which case s may be a half-integer. For any choice of s or equivalently of b, $N_1 = N_2 = 6$ is a solution of Eq. (3.11). However, this solution does not lead to a Painlevé potential but rather to a potential with logarithmic singularity which generalizes the well-known case of Ref. 8.

$$V = y^3 + 3x^2y/2.$$

Besides this formal solution the *only* other solutions of (3.12) up to the obvious permutations of N_1, N_2, N_3 are

$$N_3 = 0,\ N_1 = 0,\ N_2 = 12, \quad V = y^3 + \lambda x^3, \tag{3.13}$$

$$N_3 = \tfrac{3}{4},\ N_1 = 90,\ N_2 = 90, \quad V = y^3 + 3yx^2/16 (c = 0!), \tag{3.14}$$

$$N_3 = 2,\ N_1 = 30,\ N_2 = 30, \quad V = y^3 + yx^2/2, \tag{3.15}$$

$$N_3 = 2,\ N_1 = 20,\ N_2 = 90, \quad V = y^3 + yx^2/2 + ix^3/6\sqrt{3}. \tag{3.16}$$

Permutations of N_1, N_2, N_3 are equivalent to rotations that recover $a = 0$ where a is the coefficient of y^2x in the potential. Note that one could have, through permutations, N_1 or $N_2 = \frac{3}{4}$. In that case, expansions of both x and y contain half-integer powers of $(t - t_0)$ and superficially do not look Painlevé. However, the change of variables which recovers the Painlevé property is evident, namely the rotation back to the case $N_3 = \frac{3}{4}$ and the choice (y, x^2) instead of (y, x).

The case $N_3 = 12, N_1 = N_2 = 0$, is not completely equivalent to (3.13) because the rotation that leads from one case to another may be singular. The potential in that case writes

$$V = y^3 + 3x^2y + \mu x^3.$$

The appropriate choice of μ allows to recover all values of λ in (3.13) except $\lambda = 0$. Conversely, $\mu = \pm 2i$ do not correspond to any λ. These two conjugate potentials are not really separable. They enter in the quasiseparable class of Eq. (15) of Ref. 3. Potentials (3.14) and (3.15) have been identified by the La Jolla group.[9] The first has been integrated independently by Hall[10] and by Grammaticos, Dorizzi, and Padjen,[11] and the second has been integrated by Greene.[12]

The potential (3.16) is new. It is indeed integrable. The second integral of motion will be derived in the next subsection.

This exhausts *all* the Painlevé potentials of degree three.

B. Direct search for the integrals of motion

As we have explained in Sec. II, the integrals of motion for the potential (3.3) quartic in velocities correspond to constant f_i's. (It goes without saying that the case of general f_i's, as well as the case of cubic integrals, have been examined as well. However, we will not burden the presentation by exhibiting these calculations which did not yield any positive result.) With constant f_i's Eq. (2.10) is identically satisfied: $\ddot{x}$ and $\ddot{y}$ are quadratic in x, y. So one can integrate Eq. (2.9) for the g_i, which gives

$$\begin{aligned}
g_0 &= f_1(3y^2x + x^3/3) + (2f_2 - 2f_2b/3 + 4f_4b/3 + f_3)y^3,\\
g_1 &= f_1[3cx^3 + 3(b-1)yx^2] + f_2(2bx^3/3 + 2by^2x)\\
&\quad + f_3[-3cy^2x + (3 - b/3)y^3] - 4f_4by^2x,\\
g_2 &= 4f_4(y^4 + byx^2) + f_3(3cyx^2 + by^2x)\\
&\quad + (f_3b + 2f_2c + f_1 - f_1b)x^3.
\end{aligned} \tag{3.17}$$

Some integration constants have been taken equal to zero in (3.17), anticipating the results from the application of the compatibility condition (2.12). The latter applied to the g_i's of (3.17) leads to the following system, when one equates to zero the coefficient of each monomial $x^m y^n$:

$$\begin{aligned}
&f_3(b-3)(b-45/2) = 0,\\
&10b^2f_3 + 30cbf_2 + (45c^2 + 13b - 43b^2/3)f_1 = 0,\\
&40b^2f_4 + 30cbf_3 + 4b(b-3)f_2/3\\
&\qquad + (-90c + 60cb)f_1 = 0,\\
&-60bcf_4 + (-45c^2 + 7b^2 + 9b)f_3 + 30cbf_2\\
&\qquad + f_1(45 - 99b + 18b^2) = 0,\\
&(b-3)(112bf_4/3 + 30f_3 - 56bf_2/3) = 0.
\end{aligned} \tag{3.18}$$

The above system is linear in the f_i's but nonlinear in the coefficients b and c of the potential. The solution of this system is straightforward but quite tedious. Seven distinct (within a complex conjugation) solutions were found in all

(a) $b = 0$, c free,

(b) $b = 3$, c free.

Those two solutions correspond to the separable potential (3.13), the two forms being equivalent within a rotation of coordinates which recovers $a = 0$ in the potential. This case generalizes the usual separable form of the Hénon–Heiles potential.[7]

(c) $b = 3/16$, $c = 0$,

(d) $b = 45/2$, $c = 178\sqrt{14}i$.

One recognizes in (c) the form (3.14), while case (d) corresponds to a rotation of (3.14).

(e) $b = \frac{1}{2}$, $c = i/6\sqrt{3}$,

(f) $b = 5$, $c = 22i$,

(g) $b = 45/2$, $c = 27\sqrt{3}/2i$.

These three cases correspond to (3.16) and its possible rotations. The only case which was not recovered in this analysis and necessitated the inclusion of quadratic terms in the f_i was the potential (3.15), which corresponds to potential parameters

(h) $b = \frac{1}{2}$, $c = 0$,

or, after a rotation

(i) $b = 15/2$, $c = 7i$.

The fourth order integral of cases (a,b) and (h,i) is trivial as it is just the square of an integral quadratic in the velocities.

On the contrary case (c,d) has a genuine fourth order integral:

$$C = \dot{x}^4 - \tfrac{3}{4}yx^2\dot{x}^2 + \tfrac{1}{4}x^3\dot{x}\dot{y} - \tfrac{3}{64}x^4y^2 - \tfrac{1}{128}x^6.$$

(Here we have preferred to take $f_4 = 0$ and $f_0 \neq 0$ in order to alleviate the notations.)

The same is true for the potential

$$V = y^3 + \tfrac{1}{2}x^2y + (i/6\sqrt{3})x^3.$$

Its second integral of motion reads:

$$\begin{aligned}
C = {} & \dot{y}^4 + 2\dot{y}^2\dot{x}^2 - 2i\dot{y}\dot{x}^3/\sqrt{3} + (4y^3 + 2yx^2 - ix^3/3\sqrt{3})\dot{y}^2\\
& + (i\sqrt{3}yx^2 + x^3)\dot{x}\dot{y} + (4y^3 - 2i\sqrt{3}y^2x - ix^3/3\sqrt{3})\dot{x}^2\\
& + 4y^6 + 4y^4x^2 + ix^3y^3/3\sqrt{3}\\
& + 5y^2x^4/4 + ix^5y/6\sqrt{3} + x^6/54.
\end{aligned}$$

So every case of integrability predicted by the Painlevé analysis was indeed recovered by the direct approach for the computation of the integrals of motion.

IV. FOURTH DEGREE POLYNOMIAL POTENTIALS

As in the previous section, we start by introducing a homogeneous polynomial potential, of degree four here:

$$V = y^4 + ay^3x + by^2x^2 + cyx^3 + dx^4, \tag{4.1}$$

and we perform a rotation in order to eliminate the term y^3x, while keeping the term y^4.

In analogy to the case of cubic potentials, this turns out to be possible unless one deals with the potential

$$V = (x \pm iy)^4 + \mu(x^2 + y^2)(x \pm iy)^2. \tag{4.2}$$

This is a potential of the quasiseparable class introduced in Eq. (15) of Ref. 3. It is integrable and possesses an integral quadratic in velocities.

The potential resulting from the rotation, however, is still too general to allow for a complete investigation of the domain of integrability. So in the case of degree four potentials, we will not insist on the exhaustive nature of our analysis, contrary to what was done in Sec. III. In what follows we will limit ourselves to potentials which are parity symmetric separately in x and y, of the form

$$V = y^4 + ay^2x^2 + bx^4. \tag{4.3}$$

Before proceeding further, we remark that, in this form, the potential possesses two-well known integrable cases.[8] The first one is the usual separable case:

$$V = y^4 + bx^4, \tag{4.4}$$

which possesses an integral quadratic in the velocities. [In the special case $b = 1$, the separable potential is still of the form (4.3) after a rotation of $\pi/4$, i.e., $V = x^4 + 6x^2y^2 + y^4$.] The second case is the rotationally invariant potential:

$$V = (y^2 + x^2)^2, \tag{4.5}$$

whose second integral of motion is just the angular momentum. Actually, this potential is a member of the Darboux family we introduced in Ref. 3.

A further case of quartic potential is encountered within the new family of integrable potentials presented in Ref. 3, and which possess integrals of motion quadratic in velocities:

$$V = y^4 + \tfrac{3}{4}x^2y^2 + \tfrac{1}{16}x^4. \tag{4.6}$$

A. Painlevé analysis of the equations of motion

The equations of motion are

$$\ddot{y} = -4y^3 - 2ax^2y, \tag{4.7}$$

$$\ddot{x} = -2axy^2 - 4bx^3. \tag{4.8}$$

There are three kinds of singularities. In the first case, both x and y diverge as $(t - t_0)^{-1}$; in the second case, y diverges as $(t - t_0)^{-1}$, and x does not diverge that fast while the reverse is true in the third case.

Let us first consider the second case. Equation (4.7) fixes the coefficient of the leading term of y. In analogy with Sec. IIIA, if we define γ by

$$y = \gamma(t - t_0)^{-1} + \epsilon(t - t_0)^{-1},$$

with $\epsilon \to 0$ as $t \to t_0$, we find

$$\gamma^2 = -\tfrac{1}{2}.$$

The possible behaviors for x are determined by (4.8). Let again s be the power dependence of x. We find

$$s(s - 1) = a. \tag{4.9}$$

Here, since the equation is even separately in x and y, it would suffice to satisfy the Painlevé property for x^2 and y^2. A necessary condition for this is that s be an integer or a half-integer. A second necessary condition can be found by considering the third case of singularity described above. By direct analogy, one finds

$$a/b = u(u - 1), \tag{4.10}$$

with u an integer or a half-integer. Let us now consider the singularity where both x and y diverge as $(t - t_0)^{-1}$, i.e.,

$$x = \alpha(t - t_0)^{-1} + \epsilon_1(t - t_0)^{-1},$$

$$y = \beta(t - t_0)^{-1} + \epsilon_2(t - t_0)^{-1},$$

where ϵ_1 and ϵ_2 go to zero as $t \to t_0$, and α and β satisfy

$$2 = -4\beta^2 - 2a\alpha^2, \tag{4.11}$$

$$2 = -2a\beta^2 - 4b\alpha^2. \tag{4.12}$$

This system has in general several couples of solution (α,β). For each of them a resonance will occur whenever the determinant of the matrix M vanishes, where M is given by

$$M = \begin{pmatrix} (n-1)(n-2) + 12\beta^2 + 2a\alpha^2 & 4a\alpha\beta \\ 4a\alpha\beta & (n-1)(n-2) + 2a\beta^2 + 12b\alpha^2 \end{pmatrix}.$$

Again, n only enters through the expression $N = (n - 1)(n - 2)$, and $N = 6$ is always a solution corresponding to $n = -1$ and $n = 4$. Indeed, using (4.11) and (4.12), the matrix M becomes, with $N = 6$,

$$M = \begin{pmatrix} -4a\alpha^2 & 4a\alpha\beta \\ 4a\alpha\beta & -4a\beta^2 \end{pmatrix}.$$

The other solution N' for N satisfies

$$N' + 6 = -(12 + 2a)\beta^2 - (2a + 12b)\alpha^2. \tag{4.13}$$

A necessary condition for the Painlevé property is that N' be of the form:

$$N' = v(v - 1), \tag{4.14}$$

for every choice of the couple (α,β) that solves (4.11) and (4.12), where v is an integer that could very well depend on the couple. However, a singularity of the second or third kind with s or u half-integers, in a rotated frame, appear as a singularity of the kind considered now with v a half-integer. In order to recover them we will also accept half-integer values of v.

The problem of finding all the possible values of a and b that satisfy (4.9), (4.10), and (4.14) with N' given by (4.13) for every choice of a couple (α,β) that solves (4.11) and (4.12) is

still very complicated. In analogy with what was done in Ref. 3, we will try to find some solutions with the help of a simplifying assumption which is a slight extension of Eq. (25) of Ref. 3. This assumption is that the quantity

$$(12 + 2a)\beta^2 + (2a + 12b)\alpha^2,$$

which appears on the right-hand side of equation (4.13), is a linear combination, *as a polynomial in* α *and* β, of the quantities $4\beta^2 + 2a\alpha^2$ and $2a\beta^2 + 4b\alpha^2$ that appear in Eqs. (4.11) and (4.12), with coefficients λ and μ, which as a consequence satisfy

$$12 + 2a = 4\lambda + 2a\mu, \tag{4.15}$$

$$2a + 12b = 2a\lambda + 4b\mu. \tag{4.16}$$

Then, for any α and β, and in particular any solution of the system (4.11) and (4.12), the following will be automatically satisfied:

$$N' + 6 = 2\lambda + 2\mu.$$

The problem is now to choose a and b such that

$$2\lambda + 2\mu - 6 = v(v-1). \tag{4.17}$$

In that way, we will obtain only systems for which the resonances are the same for every choice of the couple (α,β), which is certainly not a necessary condition. Thus we don't expect this search to be exhaustive. Even the arithmetic system (4.9), (4.10), (4.14) through (4.11), (4.12), is too vast to be solved in all generality. Here are the solutions we have found:

(a) $s = \frac{1}{2}$, $u = 4$, $v = 4$, $\lambda = 0$, $\mu = 9$,

$$V = y^4 + \tfrac{3}{4}y^2x^2 + x^4/16. \tag{4.18}$$

This polynomial has been described in Ref. 3. It has a second integral quadratic in the velocities.

(b) $s = 2$, $u = 2$, $v = 2$, $\lambda + \mu = 4$.

$$V = (y^2 + x^2)^2. \tag{4.19}$$

This polynomial has also been described in Ref. 3 (for $\lambda = 0$, $\mu = 4$). It is obviously integrable.

(c) $s = \frac{1}{2}$, $u = 3$, $v = 7$, $\lambda = -9$, $\mu = 33$,

$$V = y^4 + \tfrac{3}{4}y^2x^2 + x^4/8. \tag{4.20}$$

This polynomial was not known to be integrable so far. As will be shown in the next subsection, it indeed possesses a second integral quartic in the velocities.

(d) $s = \frac{1}{2}$, $u = 17$, $v = 7/2$, $\lambda = 39/40$, $\mu = 32/5$,

$$V = y^4 + \tfrac{3}{4}y^2x^2 + 3x^4/1088. \tag{4.21}$$

Here v is a half-integer but so is s. No rotation of x and y can be found that would completely separate integers from half-integers for all singularities at the same time. Thus, there is no change of variable for which this system would be Painlevé. However, it could be "weak Painlevé" in the sense of Ref. 3.

B. Direct search for the integrals of motion

The first compatibility condition (2.10) applied on a potential of the form (4.3) gives

$$(a-2)f_3 + (2b-a)f_1 = 0. \tag{4.22}$$

We first look for solutions with the f_i's constant; f_4 can be taken equal to zero by adding to C the suitable multiple of the square of the Hamiltonian.

Provided f_i, a, and b obey (4.22), the equations for the can be integrated to give

$$g_0 = f_0(4ax^2y^2 + 4bx^4) + f_1(\tfrac{4}{3}ax^3y + 4xy^3) + f_2(2 - a/3)y$$

$$g_1 = f_0(-\tfrac{8}{3}ayx^3) + f_1[(3a-6)x^2y^2 + (3 - a/6)x^4] + f_2(\tfrac{4}{3}axy^3 + \tfrac{4}{3}ayx^3) + f_3(3 - a/6)y^4, \tag{4.2}$$

$$g_2 = f_0(\tfrac{8}{3}ax^4) + f_1(2 - a/3)x^4 + f_3(\tfrac{4}{3}axy^3 + 4bx^3y).$$

Finally, the second compatibility relation is used to obtain system of equations for the f_i's, a and b, the solution of which gives integrable quartic potentials:

$$f_1(a-6b)(a-42b) = 0,$$

$$a(a-6b)(2f_0 - f_2) = 0,$$

$$2af_3 + f_1W_1(a,b) = 0,$$

$$(\tfrac{40}{3})(a-6)f_2 + f_1W_2(a,b) = 0, \tag{4.2}$$

$$a(1 - 4a/3)f_0 + f_1W_3(a,b) = 0,$$

$$f_3(-a^2/3 + 16a - 48) + f_1W_4(a,b) = 0,$$

$$f_3(132 - 32a + 29a^2/3) + f_1W_5(a,b) = 0,$$

where the W_i's are rather complicated expressions in term a and b.

The solutions of system (4.24) is quite analogous to that of the system (3.18). One finds immediately that the system has a solution for

$$f_0 = 1, \quad f_i = 0, \quad i = 2,3,4.$$

The seven Eqs. (4.24) with the condition (4.22) reduce then

$$a(1 - 4a/3) = 0, \quad a(a-6b) = 0. \tag{4.2}$$

—The case $a = 0$ corresponds to the separable potenti (4.4), the constant of motion being the square of a quadrat constant of motion.

—The case $a = 6b$, $a = \frac{3}{4}$, i.e., $a = \frac{3}{4}$, $b = \frac{1}{8}$,

$$V = y^4 + \tfrac{3}{4}x^2y^2 + \tfrac{1}{8}x^4 \tag{4.2}$$

is just the new potential (4.20) provided by the Painlevé ana ysis. The constant of motion associated with it is written

$$C = \dot{x}^4 + (24x^2y^2 + 4x^4)\dot{x}^2 - 16x^3y\dot{x}\dot{y} + 4x^4\dot{y}^2 + 4x^8 + 16x^6y^2 + 16x^4y^4. \tag{4.2}$$

However, no trace was found of a solution associated wit the potential (4.21) at this order, even with the general for of the f_i's. This motivated an investigation of the existence a second integral of motion at order five or six in the velo ities. We will not present here any of these cumbersome ca culations. Their result is that the potential in question do not possess a second integral of motion up to order six. Th in itself, does not exclude the existence of higher order int grals. However, numerical studies of the surfaces of sectio exhibit a completely chaotic behavior, which is the signatu of nonintegrability.

V. PRECISE DEFINITION OF THE WEAK PAINLEVÉ PROPERTY

As we have seen, the nonintegrable potential (4.21) do not have the usual Painlevé property. No rotation of x and

completely separates integer powers from half-integer ones for all singularities at the same time, which would be necessary for the system to be Painlevé in terms of the rotated x^2 and y^2. However, according to the definition of Ref. 3, which was only grossly stated at that stage, this potential appears to have the weak-Painlevé property. Indeed, we said that for an integrable polynomial potential of degree $p+2$ the solutions should be expansions in terms of $(t-t_0)^{1/r}$, and we expected there r to be equal to p. The leading behavior in the neigborhood of a singularity is as $(t-t_0)^{-2/p}$, and one of the resonances is always at $2+4/p$ which makes p a reasonable candidate for the denominator of the natural power. Note, however, that, while, for p odd, there is no other candidate than p, for even p, $p/2$ is just as natural as p itself. In the case of the potential (4.21), p is equal to 2, and we do have an expansion in terms of $(t-t_0)^{1/2}$. On the other hand, for all known integrable polynomials of even degree, the expansions are really in terms of $(t-t_0)^{2/p}$. This is actually the case of the polynomials of the Darboux class[8] and also for the new class described in Ref. 3 (in terms of x^2 and y rather than x and y). Finally it is also true for the potential (4.20). Thus it appears that, for even p, whenever an expansion is found in powers of $(t-t_0)^{1/(p/2)}$, the potential is integrable. On the other hand, the case (4.21) above, which is associated with an expansion in $(t-t_0)^{1/p}$, is not integrable.

This has motivated a further investigation. We perturb potential (4.20) through the addition of a term that introduces half-integer powers. The new potential is

$$V = y^4 + \tfrac{3}{4}y^2x^2 + \tfrac{1}{8}x^4 + \lambda x. \tag{5.1}$$

The addition of the linear term excludes the existence of a second constant of the motion quartic in the velocities. Indeed it is not possible to add to the constant (4.27) terms dependent on λ so as to recover a new constant for nonvanishing λ. This does not entirely preclude the existence of a higher order integral of the motion. None was found, however up to order six. Again, the nonintegrability of this system has been confirmed by numerical studies of the surface of section which exhibit large scale chaos.

As for the Painlevé property, the resonances are, of course, not modified by the addition of a nondominant term. One can check that the new term does *not* introduce any logarithms at the resonances. The only modification is thus the introduction of half-integer powers. So we have here a second case where nonintegrability is associated with an expansion in terms of $(t-t_0)^{1/2}$ with $p=2$.

We can thus conclude with the following precise definition of the weak Painlevé property. We demand an expansion in terms of rational powers of $(t-t_0)$. The denominator of the rational exponent should be

p for odd p's,

$p/2$ for even p's,

where the degree of the polynomial potential is $p+2$. Note that this choice is rather more natural than always taking p as denominator of the natural power because this is just the denominator of the exponent of the leading behavior. Indeed the leading behavior being $(t-t_0)^{-2/p}$, the denominator is p when p is odd but is $p/2$ when p is even.

VI. CONCLUSION

In this work we have presented some new results based on the complex-plane singularity analysis of the equations of motion, using the weak Painlevé concept we introduced in Ref. 1. The dynamical systems considered correspond to the motion of a particle in a two-dimensional homogeneous polynomial potential of degree three and four. The latter limitation was imposed by the sheer complexity of the problem. Two methods have been used in parallel in order to investigate the integrability, Painlevé analysis, and direct computation of the integral. For the case of cubic interactions we have reduced the condition for the validity of the Painlevé property to a simple arithmetical equation of which we were able to exhibit all the solutions, leading to novel cases. For the case of quartic Hamiltonians, our search was not exhaustive. It led, nevertheless, to the discovery of new Painlevé potentials. The integrability was, in each case, demonstrated explicitly through the direct calculation of the integrals of motion. This search is performed systematically order by order. It cannot thus constitute a criterion of nonintegrability whenever it fails to give a positive result. However, the nonintegrability up to integrals of order six for a quartic potential which seemingly satisfies the weak Painlevé property, combined with the "nonusual" way this property is satisfied, makes the latter a serious candidate for nonintegrability. This has led us into considering in detail the notion of the "natural power," which is basic to the weak Painlevé concept.

So what emerges as a conclusion, from the results we presented in this series of papers, is that the Painlevé property is a most useful tool for the investigation of the integrability of dynamical systems, which is far from having shown the limits of its possibilities. Actually the implementation of this property to dissipative two-dimensional systems appears to be within reach.

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EXPLICIT INTEGRABILITY FOR HAMILTONIAN SYSTEMS AND THE PAINLEVE CONJECTURE

B. DORIZZI

B. GRAMMATICOS

Dépt. de Mathématiques Appliquées (MTI). Centre National d'Etudes des Télécommunications - 38-40 ave. du Général Leclerc -
92131 ISSY LES MOULINEAUX (FRANCE)

A. RAMANI

C.P.T. ECOLE POLYTECHNIQUE

91128 PALAISEAU

Abstract

We analyse a class of Hamiltonian systems in two dimensions for which we proved that a second constant of motion exists. It is shown that, using the two first integrals, the equations of motion can be written in a form which allows their integration by quadratures. An analysis of the equations of motion in this reduced form establishes the behaviour of the solutions in the complex time plane. It is shown explicitly that the systems belonging to this class possess the "weak Painlevé" property. i.e. their solutions in complex-time can present singularities of a specific algebraic type.

. A paraître dans Journal of Mathematical Physics.

In a recent work we have used the dual approach : Painlevé analysis of the equations of motion and direct search of constants of motion to study two dimensional Hamiltonian systems from the point of view of integrability [1]. Two interesting results were the fruit of this analysis. We were able to identify all the 2-D Hamiltonians which possess a second integral of motion quadratic in the velocities, generalizing the results of Darboux [2], Winternitz et al [3], Fokas et al [4]. Seven distinct classes of Hamiltonians were found in all, three of which correspond to complex potentials. The singularity analysis of the equations of motion related to these Hamiltonians revealed the following interesting feature. Although associated to an integrable system the solutions in the complex time plane did not possess the Painlevé property i.e. their movable singularities were not just poles. Rather the singularities were cuts of the type $(t-t_o)^{1/n}$.

This was in contradiction with a naïve generalisation of the ARS conjecture [5]. That conjecture associated the Painlevé property to integrability for PDE's. Our counter-example shows that it is not valid as it stands for ordinary differential systems, namely such systems can be integrable without having the Painlevé property. This has led to the formulation of the "weak Painlevé" conjecture [6].

We have thus conjectured that a two dimentional Hamiltonian system will be integrable whenever the solutions present movable singularities of the form $(t-t_o)^{1/n}$ where n is determined solely by the dominant behaviour of the system at the singularity. The exponent 1/n is called then a "natural" power and determines fully the nature of the algebraic singularities of the solutions.

The aim of the present paper is to show that for the above classes of Hamiltonians the solution of the equations of motions can be reduced to quadratures, through a proper change of variables. The singularity structure in this new system of coordinates will be examined in particular in relation to the weak Painlevé property.

I - EXPLICIT INTEGRATION OF THE EQUATIONS OF MOTION

In ref [1] we have analysed the question of integrability for Hamiltonian systems in two dimensions, and found all the systems with a second integral quadratic in the velocities. Seven distinct classes of potentials have resulted, three of which corresponded to complex potentials. In this section we will show how the two integrals of motion can be combined in order to reduce the computations of the trajectories of the system to quadratures. We will limit ourselves to the more physical case of real potentials. (The treatment of the complex ones can be performed along the same lines).

The four classes of real potentials we have identified are the following :

$$V = F(x) + G(y) \tag{1}$$

$$V = F(\rho) + \frac{1}{\rho^2}\, G(\varphi) \tag{2}$$

$$V = \frac{F(\rho + y) + G(\rho - y)}{\rho} \tag{3}$$

$$V = \frac{F(u) + G(v)}{u^2 - v^2} \tag{4}$$

where $x = \rho \sin \varphi \quad y = \rho \cos \varphi$ and $2u^2 = \rho^2 + a^2 + \sqrt{(\rho^2 + a^2)^2 - 4a^2x^2}$

and $2v^2 = \rho^2 + a^2 - \sqrt{(\rho^2 + a^2)^2 - 4a^2x^2}$

The integrals of motion associated to these systems are the following

$$C = \dot{x}^2 + 2F(x) \tag{5}$$

$$C = (\dot{x}y - \dot{y}x)^2 + 2\, G(\varphi) \tag{6}$$

$$C = \dot{x}(x\dot{y} - y\dot{x}) + \frac{(\rho + y)\, G(\rho - y) - (\rho - y)\, F(\rho + y)}{\rho} \tag{7}$$

$$C = (x\dot{y} - y\dot{x})^2 + a^2\dot{x}^2 + \frac{2[u^2F(u) - v^2\, G(v)]}{v^2 - u^2} \tag{8}$$

The integration of case (1) is immediate. The potentiel is separable in x and y. The Hamiltonian in each direction is a constant

$$E_1 = \tfrac{1}{2}\dot{x}^2 + F(x)$$

$$E_2 = \tfrac{1}{2}\dot{y}^2 + G(y)$$

and the integration for the trajectory straightforward. One thus obtains $t = t(x)$ and $t = t(y)$ which can in principle be inverted to give $x = x(t)$, $y = y(t)$.

Case (2) is also quite simple. Rewriting the Hamiltonian and the constant C in polar coordinates, we have :

$$H = \tfrac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + F(\rho) + \frac{1}{\rho^2}\, G(\varphi)$$

$$C = \rho^2\dot{\varphi}^2 + 2\, G(\varphi)$$

Combining the two constants of motion, we obtain :

$$\dot{\rho}^2 = 2E - 2F(\rho) - \frac{C}{\rho^2}$$

$$\dot{\varphi}^2 = \frac{C - 2\,G(\varphi)}{\rho^4}$$

Although this is not completly separable in the usual sense, it is still true that the variable ρ separates out. Its integration can be performed by a simple quadrature. Once $\rho(t)$ is found, the equation for φ can also be reduced to a quadrature :

$$\int \frac{d\varphi}{\sqrt{C-2G(\varphi)}} = \pm \int \frac{dt}{\rho^2(t)}$$

Case (3) can be treated in a natural way in parabolic coordinates. We put :

$$\begin{aligned} x &= 2\xi\eta \\ y &= \xi^2-\eta^2 \end{aligned} \qquad (9)$$

The two constants of motion can be written in this coordinate system as

$$\begin{aligned} E &= 2(\xi^2+\eta^2)\,(\dot{\xi}^2+\dot{\eta}^2) + \frac{F(\xi) + G(\eta)}{\xi^2+\eta^2} \\ C &= 2(\xi^2+\eta^2)\,(\dot{\xi}^2\eta^2-\dot{\eta}^2\,\xi^2) + \frac{\eta^2F(\xi) - \xi^2\,G(\eta)}{\xi^2+\eta^2} \end{aligned} \qquad (10)$$

solving for $\dot{\xi}$, $\dot{\eta}$ we readily obtain :

$$\frac{d\xi}{\sqrt{\xi^2E+C-F(\xi)}} = \pm \frac{d\eta}{\sqrt{\eta^2E-C-G(\eta)}} = \pm \frac{dt}{\sqrt{2}(\xi^2+\eta^2)} \qquad (11)$$

We thus remark that although the system is not separable in parabolic coordinates, the equation for the trajectories does indeed separate. Once this equation is integrated to yield $\xi = \xi(\eta)$, the time dependence is then also reduced to a quadrature.

Finally case (4) can be treated in elliptical coordinates. We put :

$$x = a\,\mathrm{ch}\xi\cos\eta \qquad u = a\cos\eta$$

$$y = a\,\mathrm{sh}\xi\sin\eta \qquad v = a\,\mathrm{ch}\,\xi$$

We obtain thus for the two integrals of motion :

$$E = \frac{a^2}{2}(\dot{\xi}^2 + \dot{\eta}^2)(\mathrm{sh}^2\xi + \sin^2\eta) + \frac{F(\xi) - G(\eta)}{a^2(\mathrm{sh}^2\xi + \sin^2\eta)}$$

$$C = a^4(\mathrm{sh}^2\xi + \sin^2\eta)\left[\cos^2\eta\,\dot{\xi}^2 + \mathrm{ch}^2\xi\,\dot{\eta}^2\right] + 2\frac{\left[\cos^2\eta\,F(\xi) - \mathrm{ch}^2\xi\,G(\eta\right.}{a^2(\mathrm{ch}^2\xi + \sin^2\eta}$$

In analogy to case (3) we obtain :

$$\frac{d\xi}{\sqrt{-C+2a^2E\mathrm{ch}^2\xi - 2F(\xi)}} = \pm\frac{d\eta}{\sqrt{C-2a^2E\cos^2\eta + 2G(\eta)}} = \pm\frac{dt}{a^2(\mathrm{sh}^2\xi+\sin^2\eta}$$

Once again the equation for the trajectories separates and the solutio is reduced to quadratures.

Examples :

In what follows, we present some examples of integration of the equations of motion for potentials belonging to the case (3). We start with reduced equations of motion (11). We restrict ourselves to potentials polynomial in x and y. This amounts to taking $F(\xi) = P(\xi^2)$ and $G(\eta) = -P(-\eta^2)$ where P is a polynomial. The reason for this choice is that this class is non trivial while being simpler than case (4). Several systems of physical interest belong to this class (see [1,7])and moreover the discussion of the weak Painlevé property has been (and will be) based on examples from this class.

In this paragraph, we further restrict ourselves to the case where P is a monomial, i.e. $F(\xi) = \xi^{2(m+1)}$ and $G(\eta) = (-1)^m\eta^{2(m+1)}$ $C = 0$, in order to simplify the calculations :

$$\frac{d\xi}{\xi\sqrt{E-\xi^{2m}}} = \frac{d\eta}{\eta\sqrt{E-(-1)^m\eta^{2m}}} = \frac{dt}{\sqrt{2}(\xi^2+\eta^2)} \qquad (12)$$

(with the + choice of signs in (11)). Putting $\xi^2 = \chi$, $\eta^2 = \Psi$, we get :

$$\frac{d\chi}{\chi\sqrt{E-\chi^m}} \qquad \frac{d\Psi}{\Psi\sqrt{E-(-1)^m\Psi^m}} \qquad \frac{\sqrt{2}\,dt}{(\chi+\Psi)} \tag{13}$$

A remark can be made at this point, as far as the trajectory is concerned. If $m = 2n$, one remarks that equation (12) for ξ, η and m is the same as equation (13) for χ, Ψ and n. So one obtains identical trajectories for potencials of degree n and $m = 2n$ for n even and, of course, $C = 0$.

We consider first a potential of degre $m = 2^p$ .Eq(12) can be transformed, putting $\xi^m = \chi$, $\eta^m = \psi$ to the following :

$$\frac{d\chi}{\chi\sqrt{E-\chi^2}} \qquad \frac{d\Psi}{\Psi\sqrt{E-\Psi^2}} \tag{14}$$

Integrating (14) we obtain for the trajectories

$$\Psi^2\chi^2\,(\lambda^2-1)^2 = 4E\,[\lambda\chi\Psi\,(\lambda-1)^2 - \lambda^2(\chi-\Psi)^2\,] \tag{15}$$

Note that for real χ and Ψ, the integration constant λ must be positive. If $m = 2$, $\chi = \xi^2$, $\Psi = \eta^2$, one can easily find the trajectories in terms of x and y..Eq. (15) becomes :

$$x^4\,(\lambda^2-1)^2 = 16E\,(\lambda\,x^2(\lambda-1)^2 - 4\lambda^2y^2) \tag{16}$$

Indeed, for $m = 2$ the potential is harmonic

$$V = y^2 + \frac{x^2}{4}$$

The ratio of the frequencies is two. Choosing $C = 0$ corresponds to takimg the trajectory that goes through the origin and (16) can indeed be parametrized by

$$x = A \sin\Theta$$
$$y = B \sin 2\Theta$$

i.e. the correct Lissajoux curve for ratio 2.

In a similar way one can obtain the analytic form of the trajectories in terms of x and y for m = 2p for any p. For example, for m = 4 we find.

$$\left(\frac{x^2}{4}\right)^4 (\lambda^2-1)^2 = 4E\left[\lambda\left(\frac{x^2}{4}\right)^2 (\lambda-1)^2 - \lambda^2 y^2 (x^2+y^2)\right]$$

However, explicitly integration further(12)for the time dependance is not possible in general, unless m = 2 of course.

Another case where we can give an expression for the trajectories is the Henon–Heiles potential for the choice of parameters for which J. Greene [7] has given the second constant of motion. Treating just the homogeneous cubic part of the potential, we have m = 3 in expression (12). Introducing $E = a^3$, $\chi = \frac{1}{a}\xi^2$, $\Psi = \frac{1}{a}\eta^2$, we obtain :

$$\int_{\chi}^{1} \frac{d\chi}{\chi\sqrt{1-\chi^3}} = \int_{\Psi}^{\infty} \frac{d\Psi}{\Psi\sqrt{1+\Psi^3}} + \text{const} \qquad (17)$$

The integration of (17) is straightforward but tedious. We obtain finally, the trajectory in terms of elliptic functions :

$$(u+v)(1+\alpha) + \alpha(f(u) - f(v)) - \Pi(\varphi,\gamma,k) - \Pi(\omega,\gamma,k) = c$$

where $\cos\varphi = \mathrm{cn}\, u = \dfrac{\sqrt{3}-1+\chi}{\sqrt{3}+1-\chi}$

$$\cos\omega = \mathrm{cn}\, v = \frac{\Psi+1-\sqrt{3}}{\Psi+1+\sqrt{3}}$$

$$\gamma = \frac{1}{2} + \frac{1}{\sqrt{3}}, \quad \alpha = \frac{1+\sqrt{3}}{1-\sqrt{3}}, \quad k = \frac{\sqrt{2+\sqrt{3}}}{2} \qquad (k'^2 + k^2 = 1)$$

and $f(w) = \left(\dfrac{\alpha^2-1}{k^2+k'^2\alpha^2}\right)^{1/2} \ln\left[\dfrac{\sqrt{k^2+k'^2\alpha^2}\ \mathrm{dn}\ w + \sqrt{\alpha^2-1}\ \mathrm{sn}\ w}{\sqrt{k^2+k'^2\alpha^2}\ \mathrm{dn}\ w - \sqrt{\alpha^2-1}\ \mathrm{sn}\ w}\right]$

As in the m = 2p case the integration for the time dependence does not seem possible, given the complexity of the expression for the trajectory.

III - SINGULARITY STRUCTURE OF THE EQUATIONS OF MOTION

The structure of singularities for case (c) where

$$V = \frac{F\left(\sqrt{x^2+y^2}+y\right) + G\left(\sqrt{x^2+y^2}-y\right)}{\sqrt{x^2+y^2}} \tag{18}$$

has been investigated en [1], using the A.R.S. algorithm[5], in the special case where V is a polynomial in terms of x and y, i e F (μ) is a polynomial and G (u) = - F (-u). In this section we will show directly how this structure follows from (11).

$$\frac{d\xi}{\sqrt{\xi^2 E+C-F(\xi^2)}} = \pm \frac{d\eta}{\sqrt{\eta^2 E-C+F(-\eta^2)}} = \frac{\pm\, dt}{\sqrt{2}(\xi^2+\eta^2)} \tag{19}$$

where $y = \xi^2-\eta^2$

$x = 2\xi\eta$.

The most general singularity arises when, for some value t_o of t, one of the variables, say ξ, diverges while the other does not. If F is of degree p + 3 of its argument (i e. degree 2p + 6 en ξ) then the leading singularity is of the form.

$$\xi \sim (t-t_o)^{-1/p}$$

To obtain an expansion of ξ en terms of $(t-t_o)$, we first find η in terms of ξ, then $(t-t_o)$ in terms of ξ and η (ξ) and invert. This expansion of ξ around t_o will be of the form.

$$\xi = (t-t_o)^{-1/p} \sum a_n (t-t_o)^{n/p} \tag{20}$$

similarly $$\eta = \sum b_n (t-t_o)^{n/p} \tag{21}$$

If p=1, this expansion is of Painlevé type. If p = 2, we will see later that only a_n's and b_n's for even n do not vanish and η and ξ^2 have Painlevé type expansions, although ξ does not. For other values of p,

there does not exist any power of ξ or η that exhibits the Painlevé property. Indeed ξ^p would have a pure pole only if all the a_n's vanish except those with n a multiple of p, and we will show that this is not possible.

In terms of x and y, one immediately sees that y diverges as $(t-t_o)^{-2/p}$ and x as $(t-t_o)^{-1/p}$. (Note that the degree of V in terms of x and y is p+ 2). This corresponds to one of the possible singular behaviours studied in [1], namely the only one that has the maximal dimensionality, i e, that depends on four arbitrary constants. In that paper we showed that two of these constants were t_o and the coefficient of the dominant term of x, and that the two others were related to what we called the "resonances" Here we will show directly how these four arbitrary constants follow from (19 and thus illustrate the meaning of the "resonances".

The four free constants in integating the equations of motion are E, C, and the two integration constants of (19). One will be taken as the value t_o of t where ξ diverges, the other as the value η_o of η at t_o. Where do these parameters enter in the expansion of ξ ? The first one, t_o, is bvious. Then, η_o enters because dt is proportional to $\xi^2+\eta^2$. From this one sees that η_o determines a_2 in (20).

Near t_o, at the lowest order, η behaves as

$$\eta \sim \eta_o + \quad (\eta_o{}^2 E - C + F(-\eta_o^2))\ (t-t_o)^{1+2/p}$$

Thus, E and C will first appear in the expansion of η in b_{p+2}, through the combinaison $(\eta_o^2 E - C)$. In the expansion of ξ, because of the $(\xi^2+\eta^2)$ factor in dt, will appear two orders later, namely in a_{p+4}. This is the only combination of E and C that will enter until the order $(t-t_o)^{2+4/p}$ At that order, because of the ξ^2 E term in the square root under d ξ, E will determine a $_{2p+4}$.
Indeed $\xi^2 E$ is smaller than F (ξ^2) by ξ^{2p+4}, F being of degree p+3. Consider two solutions for which ξ diverges at the same value t_o of t, that have the same value η_o for η at t_o, and different values of E and C, chosen in such a way that $\eta_o{}^2$ E-C has the same value. Then, their respective expansions for ξ will be identical up to a_{2p+3}.

In summmary, the four arbitrary constants t_o, η_o, E, C are equivalent to t_o, a_2, a_{p+4} (or b_{p+2}) and a_{2p+4}.

These are precisely the arbitrary constants which resulted from the singularity analysis in [1]. Indeed, going back to the x, y coordinates, η_o corresponds to the freedom of the coefficient of x, b_{p+2} (i e a_{p+4}) corresponds to the resonance $(1+2/p)$ in x, and a_{2p+4} corresponds to the resonance $2+4/p$. The coefficients of the polynomial F will first appear in the expansion of ξ, one after the other, through a_{2k}, $k = 2$ to $p+1$. As for the two lowest order coefficients of F, their interpretation is the following. The only significant quantity is $F(\mu) - E\mu - C$, as can be checked in (19). Changing the linear and constant terms in F just redefines E and C. One can immediatly check that the corresponding potential V is shifted accordingly.

Note that if p is even, only a_n's and b_n's with even n will be nonvanishing, and thus η and ξ^2 have expansions in terms of $(t-t_o)^{1/p'}$ where $p' = p/2$, and in this case the "natural" power as defined in [8] is indeed $1/p'$ rather than $1/p$. In particular, if $p=2$, the expansion for η and ξ^2 is indeed of Painlevé-type.

On the other hand, for $p > 2$, ξ^p does not have a Painlevé-type expansion of full dimensionality, since the arbitrary parameter a_2 would destroy this property unless it is chosen equal to zero. For all p, however, the expansions are of what we called "weak Painlevé type" [6].

Similarly one could study the case where ξ and η diverge at the same value of t_o. These solutions have a lower dimensionality, namely three, because it corresponds to choosing η_o equal to infinity. Again, one finds by direct study that the structure of the singularity in this case is exactly what has been found through the ARS algorithm in [1]. There, the lower dimentionality was associated to the negative value of one of the resonances.

$$V = [F(y+\sqrt{x^2+y^2}) + G(y+\sqrt{x^2+y^2})] \; / \; \sqrt{x^2+y^2}$$

Except for very special F's and G's the radicals will never regroup. If ever this happens as for example for

$$V = \frac{\ell n x}{\sqrt{x^2+y^2}}$$

($F(u) = G(u) = \frac{1}{2} \ell n\, u$) one would expect anyway the existence of a logarithmic singularity because the potential itself contains a logarithm. In that the case, this singularity will not suggest non-integrability.

On the other hand whenever V is such that x and y are the only "obvious" variables and moreover does not contain built-in singularity-generating terms, then integrability is conditioned by the "weak-Painlevé" property.

An other interesting question has been asked to us by A. Fokas, on whether the weak-Painlevé property was always associated to systems for which the second constant of motion is quadratic in velocities. In a future publication, we will show that this is not the case, by exhibiting several weak-Painlevé systems with non-quadratic second integrals of motion.

Finally, the above analysis will serve to shed a new light on the relationship between integrability and the analytic properties of the solutions of the equations of motion [9].

ACKNOWLEDGEMENTS

We wish to express our gratitude to M. D. Kruskal for many challenging discussions and for asking the question that this paper set up to answer.

We also acknowledge interesting discussions with P. Winternitz and A. Fokas.

IV - DISCUSSION AND OUTLOOK

It is important to realize that the free parameters in the expansions of ξ and η appear at exactly the same powers of $(t-t_o)$ as in the expansions of x and y. This illustrates the intrinsic character of the resonances. The variables ξ and η are obviously the natural ones as they lead to an explicit integrability for the equations of motions, and even in these good variables, the weak Painlevé character persists (for $p > 2$). From this, we surmise that no other change of variables will be able to restore the full Painlevé property. The answer to the question asked to us by M.D. Kruskal, namely whether the weak Painlevé character could not
by an appropriate change of variable therefore appears to be negative.

If one takes F and G as two arbitrary functions of their respective arguments, rather than two closely related polynomials, we can introduce movable singularities of any character, without destroying the integrability of the system, Still, ξ and η will obviously remain the natural variables and it is quite improbable that these movable critical singularities will be removed by an other choice of variables.

The fact that arbitrary movable singularities do not compromise integrability for two-dimensional Hamiltonian systems contradicts the weak-Painlevé conjecture. Therefore it would appear to make it useless as an integrability detector. This however, is not true. The weak-Painlevé criterion, as a heuristic tool, keeps its predictive power.

Indeed suppose we start with two arbitrary functions $F(\xi)$ and $G(\eta)$ One can thus always construct a potential V for an integrable Hamiltonian problem in terms of x and y . However, in general, the variables ξ and η will be quite "visible" in the expression of V . Therefore, a cursory glance at V will suggest the right change of variables. Indeed, for general F and G we have

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HAMILTONIANS WITH HIGH ORDER INTEGRALS AND THE "WEAK-PAINLEVÉ" CONCEPT

B. GRAMMATICOS, B. DORIZZI

Département de Mathématiques (M.T.I.)
Centre National d'Etudes de Télécommunications
38-40, rue du Général Leclerc
92131 Issy-les-Moulineaux - France

A. RAMANI

Centre de Physique Théorique
Ecole Polytechnique
91128 Palaiseau Cédex - France

ABSTRACT

We examine the singularity structure of the equations of motion associated to integrable two-dimensional Hamiltonians with second integrals of order higher than two. We show in these specific examples that the integrability is associated to a singularity expansion of the "weak-Painlevé" type. New cases of integrability are discovered, with still higher order integrals which are explicitly computed.

A553.0583 Mai 1983

I - INTRODUCTION

The singularity analysis is a most useful tool in the study of non-linear dynamical systems. The interest in this method has been kindled by the conjecture of Ablowitz, Ramani and Segur (ARS) [1] who related integrability to the singularity structure through the Painlevé property. The latter is associated to the absence of critical movable singularities on the complex time plane i.e. the singularities of the solutions are simply poles. The power of the ARS conjecture has been amply demonstrated by various works where it was shown that well-known integrable systems pass the Painlevé test. However, and what is more important, this approach has made possible the identification of new classes of integrable systems [2].

In a recent work we have examined the possibility of integrability existing independently of the Painlevé property. We thus discovered two-dimensional polynomial Hamiltonians which possess a second integral of motion, and where the equations of motion were not of Painlevé type. However they possess movable singularities of particularly simple algebraic-cut form $(t-t_0)^{1/n}$. Such a behaviour was dubbed the "weak-Painlevé" property and has been extremely useful in producing 2-D integrable Hamiltonians [3].

Once the "weakened" version of the Painlevé conjecture was proposed, two important questions have been asked by M.D. Kruskal and A. Fokas. Kruskal's question was whether the full Painlevé property (pure pole behaviour) could not be recovered from the "weak-Painlevé" on with an adequate change of variables. This question has been dealt with in detail in a previous article [4]. We have shown that the two-dimensional Hamiltonian systems exhibiting the weak-Painlevé behaviour, and which are associated to constants of motion quadratic in the velocities, are separable in some sense. The equations for the trajectory indeed separate when one expresses them in the adequate coordinate system [5]. Our anal ysis has shown that the equations of motion, even when expressed in the coordinate system which lead to separation, still are of "weak" rather than full Painlevé type. Fokas'remark at this point was that a genuine test of the "weak-Painlevé" criterion would be offered by integrable Hamiltonians which are decidedly non separable. Therefore 2-D Hamiltonians with a second integral of degree

higher than two in the velocities were investigated.

The aim of this paper is to examine, from the point of view of the singularity structure, two Hamiltonians : one due to Fokas himself [6] and one due to Holt [7], for which cubic integrals of motion are known. We will show that these Hamiltonians lead indeed to expansions of the "weak-Painlevé" type for the solutions. In the case of the Holt Hamiltonian further cases of integrability are discovered. Their second integrals are calculated. In the process of calculating these integrals we investigate the possibility of modifying the potential through additionnal terms while preserving integrability. We derive thus the additive terms for the Holt potential and for the Hénon-Heiles one (which we have analysed in a previous work [8]).

II - PAINLEVE ANALYSIS OF THE FOKAS-LAGERSTROM HAMILTONIAN

In a recent paper [6], Fokas and Lagerstrom proposed the following two-dimensional integrable Hamiltonian :

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x^2 - y^2)^{-2/3} \qquad (1)$$

for which the second integral is :

$$I = (\dot{x}^2 - \dot{y}^2)(x\dot{y} - y\dot{x}) - 4(y\dot{x} + x\dot{y})(x^2 - y^2)^{-2/3} \qquad (2)$$

The Hamiltonian assumes an even simpler expression in the variables $x \pm y$.

We rewrite H , up to scalings, as

$$H = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{3}{2}(XY)^{-2/3} \qquad (3)$$

The equations of motion read

$$\begin{aligned} \ddot{X} &= X^{-5/3} Y^{-2/3} \\ \ddot{Y} &= X^{-2/3} Y^{-5/3} \end{aligned} \qquad (4)$$

An obvious possible singular behavior is

$$X \sim Y \sim (t-t_0)^{\alpha}$$

with $\alpha - 2 = -\frac{7}{3}\alpha$ i.e. $\alpha = \frac{3}{5}$.

The leading behavior is not a pole. A priori, one could expect either of two possibilities :

- the "full Painlevé" situation, where X^5 and Y^5 have a pure triple pole ;

- the "weak Painlevé" case, where all powers of $(t-t_0)^{1/5}$ appear in the expansion.

In order to settle this question, we compute the resonances [1]. We find -1, $-\frac{4}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$. As usual, -1 is associated to the arbitrariness of t_0 and two free parameters enter at order $\frac{3}{5}$ and $\frac{4}{5}$. We are thus in a "weak-Painlevé" situation.

Because of the presence of one negative resonance besides -1, this solution depends on three free parameters instead of four, and is not generic. We must now look for the generic singular solution. Up to the exchange of x and y, this solution is obtained when Y has a divergent second derivative, but has a finite, nonzero limit A at the singularity.

The dominant part of the X equation is :

$$\ddot{X} = A^{-2/3} X^{-5/3}$$

giving the leading behavior $X \sim (t-t_0)^{3/4}$ and resonances -1 and $+\frac{1}{2}$ (this is still a "weak-Painlevé" situation).

As for Y it behaves as

$$Y = A + B(t-t_0) + C(t-t_0)^{3/2} + \ldots$$

with A and B free. The $(t-t_0)^{3/2}$ term in Y is such that its second derivative balances the most divergent term (proportional to $X^{-2/3}$) in the corresponding equation.

The four free parameters of this generic solution are t_0, A, B and the coefficient of $(t-t_0)^{5/4}$ in X (or, equivalently, of $(t-t_0)^2$ in Y) which is related to the energy.

In conclusion, both the obvious, nongeneric solution and the (far from obvious) generic one have weak-Painlevé type expansions. This would suffice in order to answer Fokas' question as to the existence of non-separable potentials with the weak-Painlevé property. However further examples can be offered.

III - PAINLEVE ANALYSIS OF THE HOLT HAMILTONIANS

In his analysis of integrable potentials in two dimensions Holt has constructed a Hamiltonian which processes a second integral cubic in the velocities [7,9]. The form of this Hamiltonian is :

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{3}{4}\lambda x^{4/3} + (y^2 - \frac{3}{2}\mu)\, x^{-2/3} \qquad (5)$$

with $\lambda = 1$ (we have introduced λ for further use).

Its second integral, still for $\lambda=1$, reads :

$$C = \dot{y}(2\dot{y}^2 + 3\dot{x}^2) + 3\dot{y}\,[2(y^2 - \tfrac{3}{2}\mu)x^{-2/3} - 3x^{4/3}] + 18\,\dot{x}y\,x^{1/3} \qquad (6)$$

The corresponding equations of motion are :

$$\ddot{x} = -\lambda x^{1/3} - \mu x^{-5/3} + \frac{2}{3}\, y^2 x^{-5/3}$$
$$\ddot{y} = -2yx^{-2/3} \qquad (7)$$

The Painlevé analysis in the case $\mu \neq 0$ is quite intricate. In fact it resembles closelythe analysis we presented for the Fokas'

Hamiltonian. That is y has a divergent second derivative but a finite nonzero limite at the singularity.

Three cases can be distinguished (in which the parameter λ does not play any role) :

i) $x \sim A\tau^{6/5}$

$y \sim \sqrt{\frac{3\mu}{2}} + B\tau^{6/5}$ $\qquad (\tau = t - t_0)$

where A, B are determined by the most diverging term in the equations of motion . The resonances in this case are $-1, -\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}$. So this is a one parameter solution.

ii) $x \sim A\tau^{9/8}$

$y \sim \sqrt{\frac{3\mu}{2}} + B\tau + C\tau^{5/4}$

where one free parameter, say B, determines A and C. We look for the resonances through a term $\gamma\tau^{5/4+n}$ in y. We find $n = -\frac{5}{4}$ which means that t_0 is free, $n = -\frac{1}{4}$ (B is free) and $n = -\frac{1}{2}, -\frac{3}{4}$, which are just formal resonances. This is a two parameter solution.

iii) $x \sim D\tau^{3/4}$

$y \sim A + B\tau + C\tau^{3/2}$

with A and B free (C and D fixed through eq. 7).We look for resonances through terms $\gamma\tau^{3/4+n}$ in x and $\delta\tau^{3/2+n}$ in y. We find $n = -1$ (due to t_0), $n = -\frac{3}{2}, -\frac{1}{2}$ (associated to A and B) and $n = \frac{1}{2}$ which is a genuine resonance (related to the energy). So the solution is the generic four parameter one and it is of the weak-Painlevé type.

We now turn to the case

$$\mu = 0,$$

for which λ will plays an essential role. In this case x diverges as $x \sim \tau^3$ while y can diverge either as $y \sim \tau^3$ or as $y \sim \tau^n$ with rational n.

In the first case ($y \sim \tau^3$) the resonance condition leads to

$$6\lambda = 6 + m(m+1) \tag{8}$$

where m must be an integer. In the second case ($y \sim \tau^n$) we obtain :

$$n(n-1) = \frac{12}{\lambda} \tag{9}$$

The only rational solutions to systems (8,9) are :

$\lambda = 1$	$m = 0$	$n = 4$
$\lambda = 2$	$m = 2$	$n = 3$
$\lambda = 6$	$m = 5$	$n = 2$
$\lambda = 16$	$m = 9$	$n = \frac{3}{2}$
$\lambda = 27$	$m = 12$	$n = \frac{4}{3}$
$\lambda = \frac{256}{3}$	$m = 22$	$n = \frac{9}{8}$

Of these solutions $\lambda = 1$ corresponds to the known Holt potential ; $\lambda = 2$ is a formal solution of (8,9) which does not satisfy the Painlevé property . Indeed , the leading behavior is not rational but logarithmic. Cases $\lambda = 6$, $\lambda = 16$ are new integrable cases which possess the full Painlevé property (for $\lambda = 16$ this is true in terms of y^2 rather than y).We have indeed been able to compute the integrals of notion of these two cases at order four and six in the velocities. The two remaning cases are, at best, genuine "weak-Painlevé" cases and candidates for integrability. However we have not been able to identify a second constant of motion, at least up to order six in the velocities.

IV - DIRECT SEARCH FOR INTEGRALS OF MOTION FOR THE HOLT POTENTIAL

In this section, we look for potentials of the form :

$$V = y^2 G_1''(x) + y G_2'(x) + G_3(x) + F(y) \tag{10}$$

for which the equations of motion admit apart from the energy a second invariant quartic in the velocities. Such potentials are a

generalization of Holt's form. Following the method exposed in [2,8], one immediately sees that because

$$V_{xy^3} = 0$$

the form of the invariant will be :

$$C = f_0\dot{x}^4 + g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h \qquad (11)$$

where f_0 is a constant and g_0, g_1, g_2, h are functions of x and y.

The expressions for the g_i' s read :

$$\begin{aligned} g_0 &= y^2G_1'' + yG_2' + G_3 \\ g_1 &= -2yG_1' - G_2 \\ g_2 &= 2G_1 \end{aligned} \qquad (12)$$

(with $4f_0 \equiv 1$)

We now write the last relation that allows the calculation of h and which will impose constraints on the nature of the arbitrary functions G_i and F.

$$g_1(V_{xx} - V_{yy}) + 2(g_2 - g_0)V_{xy} - (2g_{0,y} - g_{1,x})V_x + (2g_{2,x} - g_{1,y})V_y = 0 \qquad (13)$$

It is clear that different kinds of terms will be involved in this relation.

Let us first consider the case where F = 0 ; the only terms depending on y in this relation are of the form $y^k\,\varphi(x)$, for k a non negative integer up to three.

The preceeding relation leads thus to four distinct equations. Quite easily, one realises that G_2 has to be set to zero if $G_1''' \neq 0$ and we are thus left with the two relations :

$$\begin{aligned} &G_1'G_1'''' + 5\,G_1''G_1''' = 0 \\ &G_1'J'' + 3\,G_1''J' + 2\,G_1'''J = 0 \end{aligned} \qquad (14)$$

(with $J = G_3 - 2G_1$)

A nice algebraic solution (with $G_1'' \neq 0$) is :

$$G_1 = \frac{9}{4} x^{4/3}$$

Integration for J is quite easy and reads :

$$J = \kappa x^{2/3} + \mu x^{-2/3}$$

and the potential V has the form :

$$V = x^{-2/3} y^2 + \kappa x^{+2/3} + \mu x^{-2/3} + \frac{9}{2} x^{4/3} \qquad (15)$$

We will now introduce the function F. If F is not polynomial, it will not interfere with other terms in relation (13). G_1 and J are thus not modified. That leads to the following equation for F :

$$3F' + yF'' = 0$$

that is $F = \frac{\nu}{y^2}$.

The most general solution of that kind is thus the following

$$V = x^{-2/3} y^2 + \kappa x^{2/3} + \mu x^{-2/3} + \frac{9}{2} x^{4/3} + \frac{\nu}{y^2} \qquad (16)$$

This is in fact a generalization of the Holt potential with $\lambda = 6$ which we recover for $\kappa = \nu = 0$. However we have two extra-terms in $x^{2/3}$ and $\frac{1}{y^2}$ which are compatible with the existence of an integral of order four :

$$C = \frac{\dot{x}^4}{4} + (y^2 x^{-2/3} + \kappa x^{2/3} + \frac{9}{2} x^{4/3} + \mu x^{-2/3}) \dot{x}^2 - 6 x^{4/3} y \dot{x}\dot{y}$$
$$+ \frac{9}{2} x^{4/3} \dot{y}^2 + y^4 x^{-4/3} - 9 y^2 x^{2/3} - 2\kappa y^2 + \kappa^2 x^{4/3} + 9\kappa x^2$$
$$+ \frac{81}{4} x^{8/3} + 2 y^2 \mu x^{-4/3} + 9 \mu x^{2/3} + \mu^2 x^{-4/3} + 9\nu x^{4/3} y^{-2} \qquad (17)$$

At this point two interesting results can be obtained concerning previously studied Hamiltonians. Indeed consider the case where F is a polynomial of degree four, namely

$$F' = ay^3 + by^2 + cy + d$$

Consider first solutions where $G_2 = 0$.

The two preceeding relations for G_1 and J are transformed into :

$$G_1'G_1 + 5\,G_1''G_1 - 6\,aG_1' = 0$$
$$G_1'J'' + 2\,G_1'''J + 3\,G_1''J' - 4\,cG_1' = 0$$

and $b = d = 0$.

From the first equation, we get :

$$G_1 = \frac{\lambda}{4}x^4 \quad , \quad a = 16\,\lambda$$

from which one obtains for H :

$$J = \kappa x^2 + \mu x^{-2} + \nu x^{-6}$$

and the solution V writes :

$$V = 8y^4 + 6x^2y^2 + x^4 + \mu x^{-2} + \nu x^{-6} + \kappa(x^2 + 4y^2) \qquad (18)$$

This potential has been examined in our paper on integrable polynomial potentials, and its second integral of motion was identified. We remark here that the additional terms $\mu x^{-2} + \nu x^{-6}$ do not destroy integrability. Indeed the second integral is :

$$\begin{aligned} C = {} & \dot{x}^4 + (24\,x^2y^2 + 4x^4 + 4\nu x^{-6} + 4\mu x^{-2} + 4\kappa(x^2 + 4y^2))\dot{x}^2 \\ & - 16\,yx^3\dot{x}\dot{y} + 4x^4\dot{y}^2 + 16\,x^4y^4 + 16\,x^6y^2 + 4x^8 + 16\,\kappa y^2x^4 \\ & + 48\,\nu x^{-4}y^2 + 16\,\mu y^2 + 8\,\kappa x^6 + \frac{8\nu}{x^2} + 4\,\kappa^2x^4 + \frac{8\,\nu}{x^8} \\ & + 4\,\frac{\mu^2 + 2\kappa\nu}{x^4} + 4\frac{\nu^2}{x^{12}} + 8\,\mu x^2 \end{aligned} \qquad (19)$$

In the case $G_2 \neq 0$, similar calculations lead to the Hénon-Heiles potential which as been integrated by us [10] and, independently, by L.S. Hall [9]. The integral of motion is again quartic in the velocities and integrability is preserved despite the addition of terms of the form $\frac{\mu}{x^2} + \frac{\nu}{x^6}$. So this integrable Hénon-Heiles potential reads :

$$V = \frac{a}{2}(x^2 + 16y^2) + d(x^2y + \frac{16}{3}y^3) + \frac{\mu}{x^2} + \frac{\nu}{x^6} \qquad (20)$$

with invariant :

$$C = \frac{1}{4}\dot{x}^4 + \left[\frac{a}{2}x^2 + dx^2y + \frac{\mu}{x^2} + \frac{\nu}{x^6}\right]\dot{x}^2 - \frac{dx^3}{3}\dot{x}\dot{y} + \frac{2d\nu}{x^4}y$$
$$+ \frac{2\mu\nu}{x^8} + \frac{\nu^2}{x^{12}} - \frac{d^2x^4}{3}y^2 - \frac{d^2x^6}{18} + \frac{a^2x^4}{4} + \frac{(a\nu + \mu^2)}{x^4} + \frac{2}{3}\mu dy \qquad (21)$$

For the case $\lambda = 16$ we look for a second integral of degree six in the velocities (having verified that degree five did not lead to any interesting results). We have in general :

$$C = e_0\dot{x}^6 + e_1\dot{x}^5\dot{y} + e_2\dot{x}^4\dot{y}^2 + e_3\dot{x}^3\dot{y}^3 + e_4\dot{x}^2\dot{y}^4 + e_5\dot{x}\dot{y}^5$$
$$+ f_0\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 +$$
$$+ g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h \qquad (22)$$

with f_i's, g_i's, h functions of x, y and e_i's constants. The absence of $\dot{y}^6$ in C guarantees that C is not proportional to H^3. From the compatibility conditions [8] we find readily :

$$e_3 = e_4 = e_5 = 0$$

while some further calculations give $e_1 = 0$. The f_i's can thus be computed :

$$f_0 = 6\, e_0 y^2 x^{\alpha} + \frac{9}{2}\, e_0\, \lambda x^{\alpha+2}$$

$$f_1 = 12\, x^{\alpha+1}\, y(e_2 - 3e_0)$$

$$f_2 = 4\, e_2 y^2 x^{\alpha} + 9\, x^{\alpha+2}\, (\frac{\lambda}{3}\, e_2 - e_2 + 3e_0) \qquad (23)$$

$$f_3 = -24\, e_2 x^{\alpha+1}\, y$$

$$f_4 = 18\, e_2 x^{\alpha+2}$$

with $\alpha = -\frac{2}{3}$. For the calculation of the g_6's we have a first compatibility condition

$$(\lambda - 21)e_2 + 15\, e_0 = 0 \qquad (24)$$

and their general form is:

$$g_0 = A_0 y^4 x^{2\alpha} + B_0 y^2 x^{2\alpha+2} + C_0 x^{2\alpha+4}$$

$$g_1 = A_1 y^3 x^{2\alpha+1} + B_1 y x^{2\alpha+3} \qquad (25)$$

$$g_2 = A_2 y^4 x^{2\alpha} + B_2 y^2 x^{2\alpha+2} + C_2 x^{2\alpha+4}$$

with :

$$A_0 = 12\, e_0$$

$$B_0 = 18\, \lambda e_0 + 36(e_2 - 3e_0)$$

$$C_0 = \frac{27}{4}\, \lambda^2 e_0$$

$$A_1 = 24(e_2 - 3e_0)$$

$$B_1 = \frac{26}{5}\, (e_2 - 3e_0)(4\lambda - 9) \qquad (26)$$

$$A_2 = 4e_2$$

$$B_2 = 12(\lambda - 6)e_2$$

$$C_2 = -(e_2 - 3e_0)(\frac{351}{4}\, \lambda - \frac{243}{10}) + \frac{9}{4}\, \lambda^2 e_2$$

The last compatibility relation for the calculation of h reads :

$$(e_2 - 3e_0)(\lambda^2 - 71 - 6) = 0 \qquad (27)$$

So either $\lambda = 1$ or $\lambda = 6$ or $e_2 = 3e_0$. The first two cases were found previously : for the first the present invariant is the square of the cubic integral, while for the second we have the product of the energy times the quartic invariant. The only remaining possibility is $e_2 = 3e_0$ and replacing back into equation (24), we find $\lambda = 16$. This gives us a genuine sixth order integral of motion :

$$C = \dot{x}^6 + 3\dot{y}^2\dot{x}^4 + 6(\dot{x}^4 + 2\dot{x}^2\dot{y}^2)(y^2 + 12x^2)x^{-2/3} - 72\, x^{1/3} y\, \dot{x}\dot{y}^3 +$$
$$+ 54\, x^{4/3}\dot{y}^4 + 12(\dot{x}^2 + \dot{y}^2)(y^2 + 144x^4)x^{-4/3} + 72\, x^{2/3} y^2(4\dot{x}^2 - 5\dot{y}^2)$$
$$+ 8(y^6 - 45y^4x^2 + 432\, y^2x^4 + 1728\, x^6)x^{-2} \qquad (28)$$

We have not found additive terms for this potential. Cases $\lambda = 27$ and $\lambda = \frac{256}{3}$ do not have second integrals of motion up to degree six in the velocities.

V - DISCUSSION AND OUTLOOK

In this paper we have presented evidence that the "weak-Painlevé" concept which we have introduced in our previous work is not associated exclusively to 2-D Hamiltonians with a quadratic second constant of motion. Two integrable Hamiltonians have been analysed : one due to Fokas and Lagerstrom and one due to Holt. We have shown that the solutions of both Hamiltonians are of the "weak-Painlevé" type i.e. they present algebraic branch points of a specific type fixed by the dominant behaviour of the equations of motion. The variety of these "natural" exponents makes the existence of a transformation, through a change of variables, from "weak" type to a full Painlevé pole-type expression extremely improbable. (This richness of natural exponents would be even greater if we had included in our Painlevé analysis the additional terms discovered in section IV).

At this point a question still lingers concerning the Holt potential. Are the cases corresponding to $\lambda = 27$ and $\lambda = \frac{256}{3}$ indeed integrable ? (The resonances being at $m = 12$ and $m = 22$ respectively make the check of the compatibility condition next to impossible).

We have not been able to identify a second constant of motion up to order six in the velocities. Calculations beyond this order would necessitate extensive formal computer calculations of increasing complexity.

As a byproduct of our calculations we have obtained additional terms for the Hénon-Heiles potential as well as for the quartic polynomial potential the integrability of which we had treated in our previous work.

As a conclusion we can state once more that the singularity analysis is indeed a most powerful tool in the domain of non-linear systems.

ACKNOWLEDGEMENTS

The authors wish to thank A. Fokas for its remark which was at the origin of this paper. We also wish to thank M.D. Kruskal for stimulating discussions.

NOTE

Once the present work was completed in its present form, we were informed that J. Hietarinta has performed an analysis of the Holt potential, identifying and integrating the cases $\lambda = 1, 6, 16$ (in our notations). In the latter case he presents a simpler form for the constant by computing $I = 8H^3 - C$ where C is given by eq. 28. He finds

$$I = \dot{y}^6 + 3\dot{y}^4\dot{x}^2 + (18x^{4/3} + 6y^2x^{-2/3})\dot{y}^4 + 72x^{1/3}y\dot{y}^3\dot{x} + 648\,x^{2/3}y^2\dot{y}^2 + 648\,y^4$$

We wish to thank Dr. J. Hietarinta for making his results available to us before publication (J. Hietarinta, Turku Univ. Report FTL-R44).

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CHAPITRE 6

SYSTÈMES DISSIPATIFS

On rencontre des phénomènes dissipatifs dans des domaines variés et nombreux des sciences physiques. La modélisation de tels phénomènes non conservatifs se fait généralement au moyen de systèmes différentiels ordinaires. L'évolution dans le temps des différentes composantes du système est gouvernée par un ensemble d'équations différentielles ordinaires, couplées, non linéaires en général.

Pour des conditions initiales données, on peut, en principe, intégrer numériquement ces équations et ainsi obtenir la solution à tous les temps suivants. Cette description du phénomène en question en termes d'un système dynamique dissipatif est pleinement justifiée à un niveau macroscopique mais nettement moins à un niveau microscopique. La dissipation est le plus souvent associée à un processus stochastique, au niveau moléculaire, et ainsi à des perturbations aléatoires extérieures au système dynamique. Néanmoins, dans ce travail, nous nous concentrerons sur la description déterministe et macroscopique du système dissipatif.

Jusqu'à ces dix dernières années, la dynamique des systèmes dissipatifs était considérée comme particulièrement simple. On ne connaissait que deux exemples d'attracteurs : les points fixes (à distance finie ou à l'infini) et les cycles limites. Il est maintenant clair que la situation est plus compliquée. A côté du comportement régulier, on en observe un, irrégulier, chaotique. Cette situation a été décrite en détail dans le chapitre 2.

Dans la série de papiers qui suivent, nous ne nous concentrons pas sur le comportement chaotique des systèmes dissipatifs mais plutôt, au contraire, nous examinons les conditions sous lesquelles un tel système présente un comportement doux, régulier, qui permet les prédictions à long terme.

Un tel comportement est réalisé de manière générique à chaque fois que le système est intégrable. Mais là se pose le problème essentiel concernant les systèmes dissipatifs. Comment définir l'intégrabilité pour de tels systèmes ?

Dans le cas des systèmes hamiltoniens, une définition précise est généralement admise ; ce n'est plus le cas ici.

Il suffit pour réaliser cela de considérer le système de Lorenz qui a été initialement proposé comme un modèle hydrodynamique simple pour la description de l'instabilité de Benard :

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = -y + rx - xz$$
$$\dot{z} = -bz + xy$$

Ce système a été prouvé intégrable pour différents choix des paramètres σ, b, r :

1/ $\sigma = 0$: les équations sont linéaires.

2/ $\sigma = \frac{1}{2}$, $b = 1$, $r = 0$, dans ce cas l'intégration du système se ramène à des quadratures.

3/ $\sigma = 1$, $b = 2$, $r = \frac{1}{9}$. Par l'intermédiaire de la constante du mouvement $C = (x^2-2z)e^{2t}$, les équations se réduisent à la seconde transcendante de Painlevé.

4/ $\sigma = \frac{1}{3}$, $b = 0$, r libre. Le système s'intègre au moyen de la troisième transcendante de Painlevé.

Cet exemple, ainsi que d'autres étudiés par la suite, conduit à tenter de définir l'intégrabilité pour un système dissipatif de la façon suivante :

a/ Soit la solution des équations de mouvement se ramène à des quadratures.

b/ Soit les équations se transforment par des changements de variables locaux en un ensemble d'équations différentielles linéaires, généralement à coefficients non constants, ce qui fait que l'on ne peut pas écrire la solution sous forme de quadratures.

c/ Soit les équations se réduisent à une équation transcendante de Painlevé. Ces équations sont intégrables dans le sens où elles se ramènent à des équations linéaires par des changements de variables non locaux, c'est-à-dire que l'on est en présence d'équations intégrales linéaires.

Un outil précieux dans l'étude de l'intégrabilité est fourni par l'existence de constantes de mouvement qui peuvent être dépendantes du temps. Ces constantes permettent, en effet, de réduire le nombre des équations du système mais sans en préserver le caractère autonome.

En particulier, si un système d'ordre N possède N constantes dépendantes du temps, indépendantes, sa résolution peut en principe se ramener à un problème algébrique.

Malheureusement, contrairement à ce qui se produit dans le cas hamiltonien, il n'existe pas de méthode systématique pour chercher les constantes du mouvement étant donné qu'elles dépendent explicitement du temps d'une manière souvent peu prévisible.

Prouver qu'un système dissipatif est intégrable apparaît donc une énorme tâche et fort heureusement, le critère de Painlevé, tel qu'il a été exposé dans le chapitre 4, permet ici aussi de détecter les cas d'intégrabilité.

Néanmoins, il apparaît sur des exemples précis que les cas Painlevé ne coïncident pas avec les cas intégrables. Plus précisément, on vérifie que des cas intégrables avec la propriété de Painlevé font partie d'une famille beaucoup plus large de cas intégrables.

Nous présentons dans la suite deux publications :

La première a trait à l'étude d'un système dissipatif autonome, très simple. Il s'agit de deux équations différentielles ordinaires du premier ordre couplées à non linéarité quadratique.

Dans ce cas, s'il existe une constante du mouvement indépendante du temps, on peut déterminer les trajectoires du système, puis l'évolution dans le temps du mouvement au moyen de quadratures.

Nous présentons l'analyse des singularités du système, ainsi que les intégrales du mouvement dans les cas qui possèdent ou pas la propriété de Painlevé.

Dans la seconde publication sont présentés des modèles tridimensionnels empruntés à des domaines variés de la physique non linéaire : le modèle de Rikitake des dynamos à deux disques, qui fut initialement proposé pour la description de la variation dans le temps du champ magnétique terrestre ; les équations de Lotka Volterra pour trois espèces en compétition, qui réalisent un modèle bien connu en dynamique des populations ; le système de Rössler, qui a été proposé comme modèle de dynamique des réactions chimiques.

Dans le domaine des oscillations de plasma, ont été étudiées les interactions de trois ondes avec des non-linéarités quadratiques. Nous examinons ainsi en détail deux systèmes étudiés par Ott et Rabinovich.

Dans chacun des cas, une analyse Painlevé détaillée permet de sélectionner des cas intégrables. L'intégrabilité est alors prouvée en vérifiant que le système satisfait à l'une des conditions *a/*, *b/* ou *c/* énoncées plus haut.

Comme pour le système précédent à deux dimensions, certains systèmes peuvent être intégrables sans pour autant posséder la propriété de Painlevé. Mais pour les systèmes tridimensionnels, on peut aussi rencontrer une autre situation intermédiaire : une seule constante peut exister sans que le système ne soit complètement intégrable.

Dans certains cas, l'existence de cette constante peut être soupçonnée par le fait que les conditions de Painlevé sont partiellement satisfaites.

INTEGRABILITY AND THE PAINLEVE PROPERTY FOR LOW DIMENSIONAL SYSTEMS

A. RAMANI
Centre de Physique Théorique,
Ecole Polytechnique, F - 91128 Palaiseau, France

B. DORIZZI and B. GRAMMATICOS
Département de Mathématiques (M.T.I.), Centre National
d'Etude de Télécommunications, F - 92131 Issy les Moulineaux, France

T. BOUNTIS
Department of Mathematics and Computer Science,
Clarkson College, Postdam, N.Y. 13676, U.S.A.

ABSTRACT

We examine a system described by two first-order nonlinear differential equations from the point of view of integrability. The singularity analysis in the complex time plane is used to investigate the Painlevé property, which according to the Ablowitz - Ramani - Segur conjecture is a prerequisite for integrability for infinite dimensional systems. We show that for such low dimensional systems the Painlevé analysis, is still a most useful guide, but integrable cases also exist which do not possess the Painlevé property.

. A paraître dans Journal of Mathematical Physics.

I - INTRODUCTION

In the last few years, a new method has been developed for investigating the integrability of dynamical systems. The principle of the method is old. It is based on the relation between integrability of non-linear differential equations and the analytic behaviour of their solutions. The names of Fuchs [1], Kovalewskaya [2], Painlevé [3], and Gambier [4], have been associated with the early developments of this approach. This method has been resurrected recently by Ablowitz, Ramani and Segur(ARS) [5] in relation to their conjecture on the integrability of non-linear partial differential equations (PDE's). They have conjectured that a PDE is integrable whenever it possesses the Painlevé Property, i.e. the only movable singularities are poles. This conjecture was implemented through a powerful algorithm for the study of the singularity structure of all the reductions of the PDE's to ordinary differential equations (ODE's).

The fact that the initial studies of the past century mathematicians concerned ODE's, combined with the existence of numerous interesting physical systems described by ODE's led to a natural "tacit" extension of the ARS conjecture [5] to systems of ODE's. The works of Bountis, Segur and Vivaldi [6] as well as of Tabor et al [7] and Segur [8] have demonstrated the usefulness of the conjecture for dynamical system. They have led to the discovery of new cases of integrability for dissipative as well as for Hamiltonian systems [9]. One important question which can be formulated in relation to the conjecture concerning ODE's is whether the Painlevé property i.e. absence of movable singularities in the complex time plane "worse" than poles, is a prerequisite for integrability. Now, there certainly exist examples of integrable systems which do not possess the Painlevé property [10,11]. Such examples are mostly encountered in low dimensional systems. This remark has motivated the present study.

In this work, we explore the connection between the Painlevé property and integrability in systems of two first order ODE's of the form

$$\dot{x} = f(x,y) \quad , \quad \dot{y} = g(x,y)$$

where f,g are polynomials of second degree in x,y (see Sec. II). Such systems are known to arise in several conservative or dissipative physical model [12].

Complete integrability in this case is associated to the existence of two time-dependent (analytical, single valued) integrals of motion. Alternatively, a single time-independent integral suffices to determine the trajectories in the $x - y$ plane, and then the time dependence of the motion can be obtained through quadratures. Systems of this kind are encountered in the modelisation of dissipative physical phenomena. However we will also call completely integrable, systems which reduce to linear, non-autonomous differential equations.

For the system at hand we will show that the singularity analysis leads to cases where the Painlevé property is associated to integrability. However we have been able to show that there exist whole classes of cases which do not possess the Painlevé Property, and yet do possess one time independent integral. Sometimes, they even reduce to $\dot{x} = h(x)$, say, in which case they are, of course, immediately integrated by quadratures.

In Sec. III, the singularity analysis is presented, and the integrals of motion are obtained in Sec. IV for cases with and without the Painlevé property. The paper ends with a more extensive discussion in Sec. V on the relation between the Painlevé property and the question of integrability of dynamical systems, in general.

II - A SIMPLE NONLINEAR DIFFERENTIAL SYSTEM.

Let us consider the following system :

$$\dot{x} = -x^2 + axy + \alpha x + \beta y + \lambda$$

$$\dot{y} = -y^2 + bxy + \gamma x + \delta y + \mu \qquad (1)$$

where the dot denotes differentiation with respect to the time t .

If we limit ourselves to the higher order part

$$\dot{x} = -x^2 + axy = -x(x - ay)$$

$$\dot{y} = -y^2 + bxy = -y(y-bx) \qquad (2)$$

we see that each variable can be factorized in its respective time derivative. It is interesting to introduce here the unique (up to a multiplicative constant) extra linear combination of x and y that has the same property, namely

$$z = (1 + b)x - (a + 1)y \ .$$

Indeed

$$\dot{z} = -z(x + y) \ .$$

Note that if either $a = -1$ or $b = -1$, z is not distinct of x or y .

It should be made clear that our aim is to study the relation between integrability and the singularity structure and illustrate it through the above system, rather than study this system for its own sake. Therefore we will not attempt to study it exhaustively. In particular, we will limit ourselves to the generic case where x , y and z are all distinct i.e.

$$(a + 1)\ (b + 1) \neq 0$$

Moreover, <u>generically</u>, one can define new sets of variables (x', z') and (y'', z'') such that the reduced system (2) can be written

$$\left.\begin{aligned} \dot{x}' &= -x'^2 + ax'z' \\ \dot{z}' &= -z'^2 + c\,x'z' \end{aligned}\right\} \tag{3}$$

or, alternatively

$$\left.\begin{aligned} \dot{y}'' &= -y''^2 + b\,y''z'' \\ \dot{z}'' &= -z''^2 + c\,y''z'' \end{aligned}\right\} \tag{4}$$

where $$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1 \,, \tag{5}$$

with $$z' = -\frac{z}{a+1} \,, \quad x' = -\frac{(b+1)x}{c+1}$$

and $$z'' = +\frac{z}{b+1} \,, \quad y'' = -\frac{(a+1)y}{c+1}$$

Since we start with $(a+1)(b+1) \neq 0$, we will never get $c+1 = 0$ from (5). We will also discard the case where (5) does not define a finite c, namely $ab = 1$. Note that we will never get either $bc = 1$ or $ac = 1$, as that would correspond to either a or b infinite. With the above restrictions, the systems (2),(3) and (4) are completely equivalent. In an analogous way, the full system (1) possesses two other equivalent expressions in terms of (x', z') and (y'', z'').

System (1) can be further simplified through translations. Indeed, if $a \neq 0$ and $b \neq 0$ the terms βy and γx can be eliminated. The case $ab = 0$ will be studied separately.

Other convenient translations (which exist even if $ab = 0$) are those for which λ and μ are made to vanish. There are in general four such translations. In these frames, the origin is a fixed point. The character of this fixed point can be very different, depending on the values of $\alpha, \beta, \gamma, \delta$.

If $\alpha\delta-\beta\gamma < 0$, the origin is a saddle point [12]. If $\alpha\delta-\beta\gamma > 0$, it is unstable when $\alpha + \delta > 0$ ("unstable spiral" if moreover $(\alpha-\delta)^2 + 4\beta\gamma < 0$), while it is stable if $\alpha+\delta < 0$ ("stable spiral" if $(\alpha-\delta)^2 + 4\beta\gamma < 0$) . The most interesting case is the one where $\alpha+\delta = 0$ and $\alpha\delta - \beta\gamma > 0$ (or equivalently $(\alpha-\delta)^2 + 4\beta\gamma < 0$) . Then the two linear eigenvalues are pure imaginary. This would suggest that the origin is a center i.e., that all trajectories in a finite neighborhood of this fixed point are closed orbits. This, however is not necessarily true due to the nonlinear terms. In fact, in general, the origin will be a spiral point either stable or unstable and only exceptionally a center.

We have numerically observed all these behaviours in one particular case, namely $a = 1$, $b = 2$ and $\beta = -\alpha$. Along the $\alpha+\delta = 0$ line suggested by the linear stability analysis, we found a center only for $\gamma = 2\alpha$ which translates to a frame where β and γ vanish, giving $\delta = (-3/2)\alpha$, $\lambda = \mu = 0$. Quite remarkably, this very same set of values will be encountered again in the Painlevé analysis of the singularity behaviour.

Moving away from the $\alpha+\delta = 0$ line, we have also observed limit cycles enclosing the origin, where the linear instability is balanced by a nonlinear stability.

Thus, this quite simple nonlinear system can exhibit the full variety of orbit behaviour of two dimensional differential systems.

III - PAINLEVE ANALYSIS.

System (1) can be written as a simple second order differential equation in terms of either x or y (or z !), and one could wonder whether any of these equations can be reduced to one of the fifty canonical forms enumerated by Painlevé and Gambier [3,4]. There does not exist, however, a systematic way to ascertain that a given equation does or does not reduce to one of these canonical forms. On the other hand, there is a straightforward, if somewhat tedious, algorithm to check directly for the Painlevé property [5].

The first step of this Painlevé analysis is the research of the leading behaviours. For this system, there are three possible leading behaviours near the (arbitrary) singularity at $t = t_o$

$$x \text{ and } y \sim \frac{1}{(t - t_o)} , \; z \sim (t - t_o)^c ;$$

$$x \text{ and } z \sim \frac{1}{(t - t_o)} , \; y \sim (t - t_o)^b ;$$

$$y \text{ and } z \sim \frac{1}{(t - t_o)} , \; x \sim (t - t_o)^a ;$$

Any of these behaviours is absent if the corresponding exponent $a , b , c ,$ is less than -1. Because of (5), however, at least one of these numbers must be larger than -1. Therefore, there exists at least one possible leading behaviour. For the Painlevé property to be satisfied, one immediately sees that a necessary condition is that those of the exponents $a , b , c ,$ that are larger than -1 , be integers. (Incidentally, if a and b are equal to -1 , z and c are <u>not</u> defined, but the Painlevé property could still hold, depending on the linear and constant terms in (1). As we explained in the previous subsection, this degenerate case will not be dealt with). It is easy to find all solutions of (5) where the exponents a , b , c are either less than -1 or nonnegative integers. Up to permutations, these solutions are :

i) $a = 0 \quad b = n \quad c = -2-n$, n non negative integer

ii) $a = 1 \quad b = 2 \quad c = 5$

iii) $a = 1 \quad b = 3 \quad c = 3$

iv) $a = 2 \quad b = 2 \quad c = 2$

Note that $a = b = 1$ allows us to define z, but not c, because the equation for x in terms of x and z is

$$\dot{x} = -x z/2$$

and cannot be rewritten in the form (3) by an appropriate scaling. We will again discard this degenerate case.

Consider first the case $a = 0$. If β is zero in (1), the equation for x separates and is of Riccati type, and therefore has the Painlevé property [3, 13]. Once the equation for x is integrated, and a solution for x is chosen, the equation for y

$$\dot{y} = -y^2 + (bx + \delta)\, y + (\gamma x + \mu) \qquad (6)$$

is again of Riccati type, and the movable singularities for y (<u>at given x !</u>) are poles. However whenever x has a pole, the behaviour of y must be studied in more detail. Indeed the Riccati equation (6) could a priori exhibit a "fixed" singularity worse than a pole. However this "fixed" singularity for y is really a movable singularity of the original system (1), because the pole of x is movable. If it were worse than a pole, the original system (1) would not possess the Painlevé property.

A first necessary condition for this property to hold is just that b be an integer other than -1, which is precisely condition i), up to the permutation of b and c. A second necessary condition, which will then be sufficient, can be obtained as the vanishing of some polynomial in $\alpha, \gamma, \delta, \lambda$ and μ. This polynomial is homogeneous of degree $b+1$, provided each λ and μ is counted as a factor of degree two. (Here b is the nonnegative integer, while $c = -b-2$ is negative).

For instance, this necessary condition for $b = 0$ is obviously

$$\gamma = 0 .$$

For b strictly positive, γ can be absorbed by a translation of y, and in this frame the condition is

$$\begin{aligned} &\text{for } b = 1 \qquad \mu = 0 \\ &\text{for } b = 2 \qquad \mu(\alpha + \delta) = 0 \\ &\text{for } b = 3 \qquad \mu(2\alpha^2 + 3\alpha\delta + \delta^2 - \lambda + \mu/2) = 0 \end{aligned} \tag{7}$$

For b strictly positive, the vanishing of μ in this frame is always sufficient, but clearly not necessary (for $b \geqslant 2$) .

If β is not zero, then the system (1) does not have the Painlevé property. In fact, whenever y diverges faster than x, then the leading behaviour of y is $1/(t - t_o)$ and x behaves as $\ln(t - t_o)$. Moreover, because of the $-x^2$ term in the time derivative of x, even $\exp(x)$ is not free from logarithms.

In the three remaining cases, it is immediate to show that the reduced system, where all linear and constant terms are taken equal to zero is indeed Painlevé. We are now interested in the different choices of $\alpha, \beta, \gamma, \lambda, \mu$ that preserve this property. Since neither a nor b vanish, one can translate x and y so as to absorb β and γ . System (1) thus becomes

$$\begin{aligned} \dot{x} &= -x^2 + axy + \alpha x + \lambda \\ \dot{y} &= -y^2 + bxy + \delta y + \mu \end{aligned} \tag{8}$$

with properly redefined α, δ, λ and μ .

In case ii) it is easy to check that

$$\lambda = 0$$

is a first necessary condition for the Painlevé property. Finally, there are eight distinct cases unique up to a scaling, that preserve the Painlevé property, namely

$$\delta = -3\alpha/2 \qquad \mu = 0 \tag{9}$$

$$\delta = \alpha = 0 \qquad \mu \text{ free} \tag{10}$$

$$\delta = \alpha \qquad \mu = 0 \tag{11}$$

$$\delta = -4\alpha \qquad \mu = 0 \tag{12}$$

$$\delta = -\alpha \qquad \mu = 6\alpha^2/25 \tag{13}$$

$$\delta = -\alpha \qquad \mu = -6\alpha^2/25 \tag{14}$$

$$\delta = -8\alpha/7 \qquad \mu = 0 \tag{15}$$

$$\delta = -13\alpha/7 \qquad \mu = 0 \tag{16}$$

Once these cases are identified, one can look for the reduction to a canonical form [3 , 4 , 13] of one of the second order ordinary differential equations equivalent to this system. It is easier to do so for the equation in terms of x . We find that all these cases are reducible to equation (X) [13, p334].

$$\frac{d^2W}{dZ^2} = -W\frac{dW}{dZ} + W^3 - 12\,q\,(Z)\,W + 12\,q'(Z)$$

For cases (9) and (10) q is zero while it is a nonzero constant proportional to α^2 in case (11 - 13). Finally for cases (14 - 16), $q(Z)$ is a Weierstrass elliptic function. The relation between (W , Z) on the one hand, and (x , t) on the other is complicated and need not be explicited here.

In case iii), one can also easily check that

$$\lambda = 0$$

is again a necessary condition. We finally obtain three discrete cases (up to a scaling)

$$\delta = -5\alpha \qquad \mu = 0 \tag{17}$$

$$\delta = -\alpha \qquad \mu = 4\alpha^2/9 \tag{18}$$

$$\delta = \alpha \qquad \mu = 0 \tag{19}$$

and, in addition, a continuous family governed by

$$\delta = -2\alpha \qquad \mu \text{ free} \tag{20}$$

Since $b = c$, there exists a symmetry with respect to the exchange of y and z as far as the dominant terms are concerned, i.e. (2) and (3) have the same form. When the system is complete with linear and constant terms, this symmetry exchanges the cases (17) and (18) while cases (19) and, of course, (20) remain invariant.

Again, one can look for reductions of the second order differential equation for x. Cases (17 - 19) reduce to the canonical form (VII) [13, p334]

$$\frac{d^2W}{dZ^2} = 2W^3$$

while case (20), in general, reduces to the type (VIII)

$$\frac{d^2W}{dZ^2} = 2W^3 + \beta W + \gamma \quad .$$

Case iv) has the Painlevé property in one discrete case

$$\delta = \alpha \qquad \lambda = \mu = 0 \tag{21}$$

and one doubly continuous family

$$\delta = -\alpha \qquad \lambda \text{ free}, \quad \mu \text{ free} \tag{22}$$

Both cases (21) and (22) can be reduced to the canonical form (XXX)

$$\frac{d^2W}{dZ^2} = \frac{1}{2W}\left(\frac{dW}{dZ}\right)^2 + \frac{3W^3}{2} + 4\alpha W^2 + 2\beta W - \frac{\gamma^2}{2W} \quad .$$

Here, $a = b = c$, and (2), (3) and (4) all have the same form. Clearly cases (21) and (22) are each invariant with respect to permutations of x, y and z.

To conclude, we can remark that all the Painlevé subcases of ii), iii) and iv) can be integrated in terms of elliptic functions, but the explicit form of the solutions are usually very complicated.

IV - INTEGRATION OF THE EQUATIONS OF MOTION.

While the explicit expression of the solutions of ii) , iii) and iv) are quite involved, one can obtain rather easily integrals of the motion. Since several changes of variable must be made "en route", these integrals are not always time - independant in the original coordinates. In fact this is true for cases (11) - (19) and (21) while in cases (9) , (10) , (20) and (22) we find genuine constants of the motion. The latter are :

$$\text{case (9)} \quad x^3 y^2 (z - 3\alpha) \equiv [\dot{x} + x^2 - \alpha x]^2 [-2\dot{x} + x^2 - \alpha x] = \text{cte} \tag{23}$$

$$\text{case (10)} \quad (xy + \mu)^2 (xz + \mu) \equiv [\dot{x} + x^2 + \mu]^2 [-2\dot{x} + x^2 + \mu] = \text{cte} \tag{24}$$

$$\text{case (20)} \quad \dot{x}^2 - x^4 + 2\alpha x^3 - (\alpha^2 + \mu) x^2 = \text{cte} \tag{25}$$

$$\text{case (22)} \quad [\dot{x}^2 - x^4 + 2\alpha x^3 - (\alpha^2 + 4\mu - 2\lambda) x^2 - \lambda^2] / x = \text{cte} \tag{26}$$

We can immediately note that all these constants (and also the time-dependant integrals of the other cases) are "perturbations" of a fondamental constant of the motion for system (2), namely

$$x^{\frac{1}{a+1}} \; y^{\frac{1}{b+1}} \; z^{\frac{1}{c+1}} = \text{cte}$$

which can be rewritten as

$$x^{\frac{1-a}{1+a}} \; (\dot{x} + x^2)^{\frac{1}{b+1}} \; \left(\dot{x} - \frac{b+1}{c+1} x^2\right)^{\frac{1}{c+1}} = \text{cte} \tag{27}$$

This quantity is a genuine constant for any value of a and b (with c given by (5)), whether integer or not. Therefore, the homogeneous system (2) is <u>always</u> integrable, whether it has the Painlevé property or not. In some cases [other than (i) - (iv)] one can even solve Eq.(27) for $\dot{x} = h(x)$, whence the corresponding system would be explicitely integrable by quadratures. Although this contradicts the naïve form of the Painlevé conjecture, this is not too surprising due to the simplicity of the homogeneous system (2).

At this point the question arises, as to whether the only ways to

perturb system (2) though linear and constant terms, while preserving integrability, lead to Painlevé cases.

This turns out not to be true. There exist at least three families of integrable perturbations of (2), generically not Painlevé.

These families are, in the frame where $\beta = \gamma = 0$:

$$\text{. } \delta = -\alpha(b+1)/(a+1) \quad , \quad \lambda = \mu = 0 \ ; \ a,b,\alpha \text{ free} \tag{28}$$

$$x^{\frac{1}{a+1}} \, y^{\frac{1}{b+1}} \, [z - (b+1)\,\alpha]^{\frac{1}{c+1}} = \text{cte}$$

or, equivalently

$$x^{\frac{1-a}{1+a}} \, (\dot{x} + x^2 - \alpha x)^{\frac{1}{b+1}} \, [-(a+1)\,\dot{x} + (ab-1)\,(x^2 - \alpha x)]^{\frac{1}{c+1}} = \text{cte} \tag{29}$$

$$\text{. } a(b-1) = 2 \text{ (ie } b = c), \ \lambda = 0, \ a\delta + 2\alpha = 0 \ ; \ a,\alpha,\mu \text{ free} \tag{30}$$

$$x^{(1-a)/2a} \, (\dot{x}^2 - x^4 + 2\alpha x^3 - (\alpha^2 + \mu a^2)\, x^2) = \text{cte} \tag{31}$$

$$\text{. } a = 1 \ , \ \alpha = \delta = \lambda = 0 \ ; \ b,\mu \text{ free} \tag{32}$$

$$[xy + \mu/(b-1)]^{\frac{1}{b+1}} \, (xz + \mu)^{\frac{1}{c+1}} = \text{cte} \tag{33}$$

or, equivalently

$$[\dot{x} + x^2 + \mu/(b-1)]^2 \, [-2\dot{x} + (b-1)\,x^2 + \mu]^{b-1} = \text{cte} \tag{34}$$

Case (9) is a subcase of (28), case (10) is a subcase of (32) and case (20) is a subcase of (30). There does not seem to exist any integrable family, generically non-Painlevé, that encompasses case (22).

Incidentally, the Painlevé cases associated with integrals of motion depending on time in the original coordinates, can sometimes be embedded in a continuous family, generically non-Painlevé, for which there exists one such integral. Since this integral is unique and time-dependent, this does not suffices for integrability in general. To our knowledge, the Painlevé subcases are the only integrable ones.

One could wonder at this point whether system (1) is not always integrable. Numerical studies cannot easily settle the question because, in two dimensions, chaos cannot be present. Although a limit cycle is usually an indication of non-integrability, its existence does not preclude integrability, as it could be associated to a time-dependent integral. The best indication of non-integrability is perhaps the presence of a nonlinear spiral point where the linear analysis suggested a center. And indeed, as we noted in Sec. II, for the one dimensional family (up to a scaling)

$$a = 1 \ , \ b = 2 \ , \ \lambda = \mu = 0 \ , \ \beta = -\alpha \ , \ \alpha + \delta = 0 \ , \ \gamma \text{ free},$$

we obtained a center only for $\gamma = 2\alpha$, which was the only case for which we found an integral (it translates to case (9)). We can thus surmise that system (1) is not generically integrable. Still, the class of integrable cases is larger than the class identified by the Painlevé property.

Finally, consider case i), with $\beta = 0$. The first equation, in terms of x only, is easy to solve. Substituting the solution $x(t)$ into Eq. (6), we obtain a Riccati equation for y . This equation can be reduced to a linear second order equation, for all values of the other parameters, whether it possesses the Painlevé property or not. Conversely, the cases that do have this property are not any "more integrable" than the others, i.e. their solutions will generally <u>not</u> be expressible in terms of quadratures.

V - RELATION BETWEEN PAINLEVE PROPERTY AND INTEGRABILITY.

As we have seen in the previous section, the Painlevé analysis has revealed several integrable cases. For some of them (14-16), the integrals have so intricate an expression that they could not have been obtained by a pure guess-work as may be done with (29), (31) and (33). Thus, the Painlevé criterion is a most useful integrability detector. However, as was explained in the previous section, there exist whole classes of integrable cases which, generically, do not possess the Painlevé property. This is not astonishing. One degree of freedom Hamiltonian systems are integrable by a simple quadrature. First order autonomous differential equations

$$\dot{x} = f(x)$$

are also trivially integrable whatever their singularities. When the equation in question is non-autonomous, integrability is not automatic. However, one can always find integrable systems which do not satisfy the Painlevé criterion (see Appendix). So it is clear that the Painlevé approach would be an over-restrictive condition for integrability, for such systems.

The case of two degree of freedom Hamiltonian systems has been dealt with in detail, in previous publications [10,11]. Our conclusion was that whenever such a Hamiltonian system is integrable without satisfying the Painlevé criterion, then either the undesirable singularities are obvious in the structure of the Hamiltonian itself, or, when this is not the case, the weak Painlevé property [9] is satisfied. In this paper, we have treated two dimensional systems described by two first-order autonomous ODE's. Our analysis resulted in the discovery of integrable systems, quite similar in structure, some of which were Painlevé and some not. Conversely, for very similar non-Painlevé systems, some were integrable and some not. So the Painlevé approach, though undeniably fruitful, does not offer for these simple systems a clear-cut distinction between integrable and non-integrable systems.

On the other hand, the concept of integrability needs to be more clearly defined. Consider, for instance second-order non-autonomous ODE's. Painlevé [3] and Gambier [4] have analyzed these equations from the point of view of the singularity structure. Their analysis resulted in fifty equations having no movable singularities but poles. Most, but not all, of them are integrable in the classical sense : the solution can be given through quadratures. Some (type V [13, p 344], or Riccati) can be reduced to linear non-autonomous equations. The solution of the latter generally cannot be written in terms of quadratures. Finally, six are integrable only inasmuch as their solutions are the new transcendents introduced by Painlevé. At this point, the definition of integrability may appear somewhat circular. However, the introduction of the Painlevé transcendents as new "special functions" is precisely justified by the regularity of their structure in the whole complex plane. This point was brought to our attention by M.D. Kruskal. A posteriori, the particular usefulness of these transcendents, due to their frequent appearance in equations of nonlinear physics, illustrates the power of Painlevé's intuition.

For higher order non-Hamiltonian systems, very few results exist. Our analysis of several examples of third order, coupled, nonlinear autonomous ODE's [14] has identified several new integrable systems possessing the Painlevé property. We have also found several cases, which are integrable, without satisfying the Painlevé conditions [14]. However, these cases turn out to have a simple structure and their integrability can be assessed at a glance. When one goes to still higher orders the relation between the Painlevé property appears to become more intimate [14 , 15]. However, one must bear in mind that it may turn out that new transcendents are needed for the integrability of higher order systems, just like the six ones introduced by Painlevé at order two [16]. So far, no confirmed example of such a new transcendent is known. However, some good candidates can be presented. Consider the modified Korteweg de Vries (m KdV) hierarchy related to the Korteweg de Vries hierarchy by the Miura transformation [17]. The n - th equation of the m KdV hierarchy is a PDE of order $2n + 1$, which reduce to a non-autonomous ODE of the same order through the ansatz

$$u(x,t) = t^{-1/(2n+1)}\, U(\xi) \qquad \text{with} \quad \xi = x\, t^{-1/(2n+1)}$$

This equation can be integrated once, leading to a non-autonomous ODE of order $2n$. It has been shown [18] that the singularities with algebraic leading order of the three first members of this hierarchy are indeed pure poles. Although this is not a full proof of the Painlevé property, the ARS conjecture [5] asserts that all of those ODE's have only movable poles. The lowest member, in fact, is nothing else but the second Painlevé transcendental equation. It is tempting to surmise that these equations could lead to new transcendents.

In Hamiltonian systems of N degrees of freedom ($2N$ dimensional phase space) the concept of complete integrability is defined by the existence of N independent, analytic, single valued, global integrals in involution [19]. As mentioned earlier, several such integrable systems have already been identified by the Painlevé property [6 , 10]. Now, it does happen sometimes that upon integrating such a system by quadratures, one arrives at one - degree of freedom motion, which is plagued by the " uglier " singularities imaginable [20]. It has been our experience, however, that the " uglier" the singularities, the <u>easier</u> it is to see what variable transformations are needed to uncouple the degrees of freedom and integrate the equations completely [11].

Concerning infinite dimensional systems described by PDE's, it appears that the Painlevé property is indeed a prerequisite for integrability. For some very simple example integrability results from a reduction to a linear PDE by a local change in the dependent variable. For instance, the Burger's equation

$$U_t + U_{xx} + 2UU_x = 0$$

becomes $V_t + V_{xx} = 0$ with $U = \frac{V_x}{V}$,

or the Liouville equation

$$U_{xt} = \exp(U)$$

reduces to $\varphi_{xt} = 0$ with $U = \ln(2\varphi_x\varphi_t/\varphi^2)$.

However, for the vast majority of the so-called "integrable" nonlinear PDE's, integrability is associated with the Inverse Scattering Transform (IST) [21]. In this case the solutions are obtained through linear integral equations, the kernel of which obeys a linear partial differential equation. So integrability for PDE's is related to the existence of some linear problem which is sometimes simple and sometimes of considerable complexity [18].

Incidentally, these PDE's are related by similarity transformations to ODE's of the Painlevé type [5]. Thus the integrable character of Painlevé's equations as well as of the equations of the hierarchy mentioned above is once more exemplified by the fact that they can now be linearized in terms of linear integral equations [21].

VI - CONCLUSION.

We have examined in this paper, on a second order autonomous differential system, the relation between integrability and the (movable) singularities of the solutions in the complex t - plane. We have found, for this system, that integrability can exist independent of the Painlevé property. However, we have shown that the singularity analysis offers a very useful and direct method for identifying integrable systems, which would otherwise be very difficult to discover. This appears to be all the more true for systems of higher dimensionality, where there is evidence that there are deeper connections between integrability and the Painlevé property. This is in perfect agreement with the ARS conjecture [5] for infinite dimensional systems (i.e. PDE's).

ACKNOWLEDGEMENTS

The authors want to thank M.D. Kruskal for numerous stimulating discussions.

APPENDIX

Consider the obviously integrable equation

$$\frac{dx}{dt} = x^3 + ax + b \tag{A1}$$

where a and b are constants. This equation never has the Painlevé property in terms of x. In terms of x^2, it does not have this property either, if $b \neq 0$. If we make the following transformation

$$\begin{aligned} x &= y\psi(T) + z(T) , \\ t &= \varphi(T) \quad \text{with} \quad \frac{d\varphi}{dT} = \frac{1}{\varphi^2} , \end{aligned} \tag{A2}$$

Eq (A1) becomes

$$\frac{dy}{dT} = y^3 + A(T)\, y^2 + B(T)\, y + C(T) . \tag{A3}$$

A, B and C are given by

$$\begin{aligned} A &= 3u \\ B &= 3u^2 + av^2 + v'/v \\ C &= u^3 + auv^2 + bv^3 + v'u/v - u' \end{aligned} \tag{A4}$$

where $u = z/\psi$, $v = 1/\psi$,

For general A, B and C, Eq (A3) is <u>not</u> integrable. Whether A,B,C are given by (A1) or not, Eq (A3) never has the Painlevé property. Still, it is integrable when A, B and C are given by (A4). Of course, on the view of (A4) as it stands, one could guess that u and v can be used to change back to the "right" variables. However, when definite expressions for A, B, C, are given, for example

$$A = T + 3$$

$$B = 2T + 3 + \frac{1}{T}$$

$$C = T + 1 + \frac{1}{T}$$

$$\left(u = 1 + \frac{T}{3} , \; v = T , \; a = -\frac{1}{3} , \; b = 2/27\right)$$

it is not obvious anymore that a change of variables of the form (A2) is adequate. Thus, this integrable equation cannot be distinguished at a glance from similar non integrable equations, and would not be singled out by a Painlevé analysis either. For such systems, integrability is not related to the analytic properties of the solutions.

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ON THE COMPLETE AND PARTIAL INTEGRABILITY OF NON-HAMILTONIAN SYSTEMS

T.C. BOUNTIS
Department of Mathematics and Computer Science
Clarkson College, Postdam N.Y. 13676

A. RAMANI
Centre de Physique Théorique de l'Ecole Polytechnique
Plateau de Palaiseau - 91128 Palaiseau - Cedex - France

B. GRAMMATICOS and B. DORIZZI
C.N.E.T. Dept. de Mathématiques (M.T.I)
38-40 Rue du Général Leclerc - 92131 Issy les Moulineaux

ABSTRACT

The methods of singularity analysis are applied to several third order non-Hamiltonian systems of physical significance including the Lotka-Volterra equations, the three-wave interaction and the Rikitake dynamo model. Complete integrability is defined and new completely integrable systems are discovered by means of the Painlevé property. In all these cases we obtain integrals, which reduce the equations either to a final quadrature or to an irreducible second order ordinary differential equation (ODE) solved by Painlevé transcendents. Relaxing the Painlevé property we find many partially integrable cases whose movable singularities are poles at leading order, with $\ell n(t-t_o)$ terms entering at higher orders. In an Nth order, generalized Rössler model a precise relation is established between the partial fulfillment of the Painlevé conditions and the existence of N-2 integrals of the motion.

A572.0983

Septembre 1983

I - INTRODUCTION

Dynamical systems described by coupled, nonlinear ordinary differential equations (ODE's) are a common occurence in many branches of applied science [1]. These equations are generally non-Hamiltonian (i.e. not derived from a Hamiltonian function) and describe the time evolution of physical processes which are mainly dissipative in character [2]. During such processes phase space volumes "contract" and the motion is often attracted (as $t \to \pm\infty$) by fixed points or periodic orbits. However, as it is by now widely recognized, the overwhelming majority of nonlinear systems have ranges of parameter values for which the solution approach a much more exotic type of attractors : These are subsets of phase space with a Cantor-like structure, called strange attractors [3,4] on which the motion is wildly chaotic in the sense that it depends sensitively on the choice of initial conditions.

Our purpose in this paper is to find conditions for the absence of strange attractors and chaotic behavior by looking for parameter values for which the equations of motion can be completely (or partially) integrated. Consider, for example, the famous Lorenz-equations

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -y + \rho x - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{1.1}$$

[$\dot{(\ )} \equiv d(\ )/dt$], which arise in simple models of hydrodynamic turbulence [3,4]. It is well-known that for certain ranges of σ,ρ and b the solutions of (1.1) are numerically observed to exhibit wildly chaotic

behavior of the type mentioned above [3,4]. For these parameter values the analytical solutions of (1.1) are not available and are, in fact, expected to be very complicated functions of time, with branch point singularities in the complex t-plane, etc. [5].

One might ask whether there are any values of σ,ρ and b for which the Lorenz equations can be explicitly solved in terms of known mathematical functions, which are analytic (modulo poles) in the complex t-plane. In that case, one of course would expect no strange attractors or chaotic behavior to be present. This question was, in fact, asked (and answered affirmatively) by H. Segur [6], who applied an idea introduced by S. Kowalevskaya [7] in her investigations of integrable cases of the rotating rigid body : In particular, Segur set out to determine for which σ,ρ and b the solutions of (1.1) would have only poles as their movable (i.e. initial condition dependent) singularities in the complex time variable $\tau \equiv t-t_o$ (t_o is the arbitrary location of the singularity). In all these cases, he was able to show that the system was integrable in a sense that will be described below.

In this paper our aim is three-fold : First we wish to extend Segur's results to many other third order systems like (1.1) with quadratic nonlinearities. These systems are all of physical relevance and are known to exhibit chaotic behavior for some values of their parameters. What we will do is find parameter values for which the equations can be integrated explicitly, thereby having solutions which behave in a regular and predictable way. The main virtues of this approach is that it is direct, straightforward and can apply to o.d.e.'s of practically any order !

Our second goal is to discuss the notion of integrability of non-Hamiltonian systems. In particular, we will define complete integrability and show how our definition is suggested by the singularity analysis, which thus becomes a very efficient tool for identifying completely integrable systems. Finally, by fulfilling only part of the required integrability conditions, we will show, with many examples, how one can used singularity analysis to identify partially integrable systems. We emphasize that our approach is non-rigourous. It does represent, however, a step towards a better understanding of integrability in non-Hamiltonian systems.

Let us return now to the Lorenz system (1.1), which is used here as an illustration of the ideas to be exploited in this paper : Segur's result [6] was that the solutions of (1.1) possess only movable poles (and are therefore expected to be easier to solve analytically) in exactly 4 cases :

(i) $\sigma = 0$: This case is trivially solvable since the equations (4.1) become linear.

(ii) $\sigma = 1/2$, $\rho = 0$, $b = 1$: There are two time dependent integrals here :

$$y^2+z^2 = C_1 e^{-2t} \quad , \quad x^2-z = C_2 e^{-t} \tag{1.2}$$

where C_1 and C_2 are two arbitrary integration constants. Using (1.2) the system (1.1) can be reduced to a single integration (quadrature) which is completed in terms of elliptic functions.

(iii) $\sigma = 1$, $\rho = 1/9$, $b = 2$: A first integral is $x^2-2z = Ce^{-2t}$ and after a change of variables one arrives at the second of six irreducible, second order o.d.e's discovered by Painlevé and coworkers, some

sixty years ago [8,9] . They made an exhaustive list of 50 second order o.d.e's whose solutions have only movable poles ; 44 of them were, in fact, solvable by elementary functions while for the first six, new transcendental functions had to be introduced - the so called Painlevé transcendents. Thus, we shall henceforth say that a system of N first order o.d.e's has the Painlevé property iff it has only movable poles with Laurent series expansions containing $N-1$ (t_o is the Nth) free constants to be fixed by the initial conditions.

(iv) $\sigma = 1/3$, $b = 0$ (any ρ) : Here one can combine eqs. (1.1) into a single, third order equation for x , which can be integrated once to yield

$$x\ddot{x} - \dot{x}^2 + x^4/4 = Ce^{-4t/3} \quad . \tag{1.3}$$

One simple change of variables - of the type we will often perform in later sections - suffices to transform (1.3) to the third Painlevé transcendental equation [8,9].

For more details on the results of singularity analysis on the Lorenz equations the reader is referred to [10]. Moreover, several papers have already appeared in the literature on the identification of integrable systems using the Painlevé property [11-19].

At this point one should attempt to give a more or less precise definition of integrability. A system (in general, non-Hamiltonian) will be called integrable whenever one of the three following conditions is met

a) the equations of motion reduce, through the existence of integrals (like (1.2) for instance), to quadratures, as in the case (ii) of the Lorenz system described above.

b) the equations can be transformed through local changes of variables to a set of linear ODE's (like case (i) above), the coefficients of which are functions of the independent variables (even though the solutions, in general, cannot be written in terms of quadratures)

c) the equations can be integrated to an irreducible Painlevé transcendental equation as in the cases (iii) and (iv) of the Lorenz system. As we have explained in [19] these equations are integrable in as much as they can be reduced to linear equations through nonlocal changes of variables, namely linear integral equations.

In sections II and III of the present paper we present many more examples of completely integrable non-Hamiltonian systems identified by the Painlevé property. However, we wish to go here one step further and discuss the possible connections between partial fulfilment of the Painlevé property and partial integrability of non-Hamiltonian systems.

Consider, for example, the case $b = 2\sigma$ in (1.1). This condition is needed (with $b \neq 0$) to ensure that a second free constant (the first one is t_o) enters in the asymptotic expansions of the solutions with integer powers of $\tau(\equiv t-t_o)$ [6,10], cf. the Lorenz cases (ii) and (iii) above. Interestingly enough, with $b = 2\sigma$ (and any ρ) equations (1.1) have one integral ($b = 2\sigma$) :

$$x^2 - 2\sigma z = Ce^{-2\sigma t} \tag{1.4}$$

which implies (for $\sigma > 0$, say) that as $t \to \infty$ all motion tends to a paraboloidal surface, $z = x^2/2\sigma$, on which, of course, there will be no chaos, strange attractors etc. We call this case partially integrable, because no other integrals like (1.4) appear to exist.

Another partially integrable case of the Lorenz equations is $b = 1$, $\rho = 0$ (and any σ), suggested by the completely integrable system (ii), mentioned earlier. Here also one integral exists, ($b=1$, $\rho=0$) :

$$y^2+z^2 = Ce^{-2t} \qquad (1.5)$$

cf. (1.2), but no further integration appears possible. What these two partially integrable cases (1.4) and (1.5) have in common is that all their singularities are poles to lowest order with $\ell n(t-t_o)$ terms arising at higher orders.

In sections III and V of this paper, we give many more examples of this kind of partial integrability. Moreover, in section IV we show on an Nth order generalization of the Rössler system [3] that there exists a precise relation between the conditions for pole-like singularities and the existence of N-2 integrals !

Our results on partial integrability are interesting in view of the fact that similar connections have been observed between logarithmic singularities and chaotic behavior in Hamiltonian systems [20,21]. However, since we are not yet able to prove any of our statements, we appeal to examples for supporting evidence.

As for complete integrability of non-Hamiltonian systems, proofs are needed there also. Possible extensions of algebraic integrability theorems for Hamiltonian systems [16,17] or the convergence of Arnold's normal forms [20,21] are currently under investigation. Of course, the problem of integrating the equations of motion of a

dynamical system, is still very much an open one. We do believe, however, that the methods of singularity analysis offer a promising avenue, by which this very important problem can be attacked.

II - THE 3-SPECIES LOTKA-VOLTERRA MODEL

We begin this section by demonstrating the usefulness of the singularity analysis on the third order Lotka Volterra system [24]

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 + C x_1 x_2 + x_1 x_3 \\ \dot{x}_2 &= \mu x_2 + A x_2 x_3 + x_2 x_1 \\ \dot{x}_3 &= \nu x_3 + B x_3 x_1 + x_3 x_2 \end{aligned} \tag{2.1}$$

where all other possible coupling constants have been scaled away. In particular, we seek to determine all values of $\lambda, \mu, \nu, A, B, C$ for which (2.1) possesses the Painlevé property and is therefore expected to be completely integrable, in the sense described in section 1. Thus, in this regard, this section reviews certain recent results published elsewhere [18]. However, we also present here some new completely integrable cases of (2.1), which are not identified by the Painlevé property, and compare with an analogous situation encountered in second order systems of ODE's with quadratic nonlinearities [19].

Following the analysis of [18], we recall that the case of equal "growth rates", i.e. $\lambda = \mu = \nu$, is a necessary condition for (2.1) to have the Painlevé property. Moreover, scaling t so that $\lambda = 1$ $(= \mu = \nu)$ and introducing the new variables

$$x \equiv x_1 e^{-t}, \quad y \equiv x_2 e^{-t}, \quad z \equiv x_3 e^{-t}; \quad t' \equiv e^t \tag{2.2}$$

(2.1) becomes (after dropping primes on t)

$$\dot{x} = C\,xy + xz$$

$$\dot{y} = A\,yz + yx \qquad (2.3)$$

$$\dot{z} = B\,xz + zy$$

Examining the leading order behavior of the solutions of (2.3) near a movable singularity $t = t_o$ one distinguishes two possible singularity types :

(1) x,y,z all diverge as τ^{-1} $(\tau \equiv t-t_o)$

(2) Two diverge as τ^{-1} and the third behaves like τ^p $(p > -1)$ at leading order. Writing

$$x \sim B_1\,\tau^p \text{ , with } y \sim \tau^{-1} \text{ , } z \sim \tau^{-1} \text{ ,}$$

and $y \sim B_2\,\tau^q$, $z \sim B_3\,\tau^s$ respectively in the other two cases, one finds immediately upon substitution in (2.3) :

$$p = -C - \frac{1}{A} \text{ , } q = -A - \frac{1}{B} \text{ , } s = -B - \frac{1}{C} \qquad (2.4)$$

whereas the B_k is in each case $(k = 1,2,3)$ the second free constant of the asymptotic expansions (the first one is t_o).

The next necessary condition for having the Painlevé property is that only integer powers of τ be allowed in the series solutions of (2.3) near $t = t_o$. Take, for example, singularity type (1) above. Calculating higher order terms in the Laurent expansions

$$x = \frac{A_1}{\tau} + \sum_{r\geqslant 0} a_r\,\tau^r \text{ , } y = \frac{A_2}{\tau} + \sum_{r\geqslant 0} b_r\,\tau^r \text{ , } z = \frac{A_3}{\tau} + \sum_{r\geqslant 0} c_r\,\tau^r \qquad (2.4\text{ bis})$$

($A_k \neq 0$, k = 1,2,3) amounts to solving, at each order, three linear equations for a_r , b_r , c_r . The condition that the coefficient matrix of these equations vanish is

$$(r+1)\ [r^2-r-A_1A_2A_3(1+ABC)] = 0 \tag{2.5}$$

This gives the values of the resonances. At this point, one of two things can happen :

(a) 1+ABC = 0 : The free constants enter at r = -1 (this corresponds to the arbitrariness of t_o) and at r = 0, -1. Now r=0 implies that one of the A_k's is arbitrary. Checking the compatibility condition of the leading order equations we find that a solution exists iff we also have A = 1/(1-C) and B = (C-1)/C with C arbitrary. It is easy to see, after some scaling, that this reduces to the antisymmetric case already integrated completely - even for the N-species extension of (2.3) - in an earlier publication [18]. It is important to note that in this case the second singularity type (2) is actually identical to (1), as p=q=s=-1 in (2.4).

(b) 1+ABC ≠ 0 : Here both singularity types must be separately examined . The Painlevé property requires first that the roots of the quadratic equation in the square brackets in (2.5) and p,q,s in (2.4) all be integers. This implies that

$$M \equiv A_1A_2A_3(1+ABC) = m(m-1) \tag{2.6}$$

for integer m , whence the values of p,q,s can be deduced from the relation

$$\frac{1}{M} = \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{s+1} - 1 \tag{2.7}$$

All possible values of p,q,s satisfying the above requirements are listed in Table 1 below :

p	q	s	m
1	1	1	2
1	2	2	3
1	2	3	4
1	2	4	6

TABLE 1

The second condition of the Painlevé property i.e. that the asymptotic expansions possess 3 free parameters near at least one singularity type is thus verified for both cases (a) and (b) above. Any other values of A,B,C, not obtained by this analysis, will introduce logarithmic (or algebraic) branch point singularities in (2.4) and destroy the Painlevé property.

This is not to say, however, that other completely integrable cases of (2.1) without the Painlevé property do not exist ; far from it. Consider for example case (a), 1+ABC = 0 : Two simple integrals can always be found :

$$x-Cy-z/B = D \ , \ xy^{1/A} z^{-C} = E \qquad (2.8)$$

(where D and E are arbitrary constants) in terms of which we can

completely integrate the system by quadratures. In fact, we have encountered this case in an earlier paper [19] : Using the first integral of (2.8), (2.3) can be reduced to a second order system, which is identical to eq. (8) of [19].

Returning to Table 1, we recall that the first case, $p=q=s=1$, has $B = -1/(A+1)$, $C = -1-1/A$ (A free) and can be shown to be completely integrable by reducing it to a single second order, linear ODE with variable coefficients [18]. In the special case $A=1$ ($B = -1/2$, $C = -2$) the solution can, in fact, be obtained explicitly in terms of elliptic functions. The integration in this case also alerts us to further integrable cases not having the Painlevé property : Taking $A=1$ and $BC=1$, with C arbitrary, one finds two integrals

$$(x-Cy)^2 z^C/xy = D^2 , \tag{2.9a}$$

and

$$D(x-Cy) - C \int \frac{D^2 dz}{\sqrt{D^2+4Cz^C}} = E . \tag{2.9b}$$

It is interesting to note here that there are certain values of C, for which the integration in (2.9b) can be performed and the solutions explicitly obtained. They can also be identified by (non-trivially) extending the Painlevé conditions to include all powers of $(t-t_o)^{1/2}$ - the so-called "weak-Painlevé" property [25-26]. These C values are $C = -4,-1,1,4$ and the fractional power $\tau^{1/2}$ is "naturally" suggested by the examination of the r,p,q,s values at which free constants enter in the asymptotic expansions, cf. (2.5)-(2.7).

Finally, the integral in (2.9b) can also be carried out in one case which admits logarithmic singularities : C=2 (A=1 , B=1/2). The results is

$$\sqrt{2}(x-2y) = E' + D \sinh^{-1} (2\sqrt{2}\, z/D)$$

and the problem is completely integrable since it can now be reduced (with the aid of (2.9a)) to a single quadrature. The singularities, however, are logarithmic since already at leading order

$$x \sim -1/\tau \ , \ y \sim -1/2\tau \ , \ z \sim F/\tau \ \ell n\tau$$

with F arbitrary.

III - A 3-WAVE PROBLEM AND THE RIKITAKE MODEL

In this section, two more systems of 3 first order ODE's are examined by the singularity analysis. The emphasis is not only on the identification of completely integrable cases but also on the introduction of logarithmic terms by the violation of higher order compatibility conditions. Thus, relaxing the Painlevé property in this specific manner, we will discover several partially integrable cases, whose solutions have singularities which behave like poles in the limit $t \to t_o$.

A. The Reduced 3-Wave Interaction Problem

As a first example, we turn to the problem of three interacting quasi-synchronous waves in a plasma with quadratic nonlinearities [2]

$$\begin{aligned} \dot{x} &= \gamma x + \delta y + z - 2y^2 \\ \dot{y} &= \gamma y - \delta x + 2xy \\ \dot{z} &= -2z - 2zx \end{aligned} \tag{3.1}$$

Here x,y,z are proportional to the magnitude of the amplitudes of the three waves respectively, δ measures the detuning from synchronism, and the other linear terms describe effects of dissipation and the pumping of external energy (e.g. plasma heating).

The first step of the analysis is to determine all possible singularity types of (3.1). Thus, one easily finds that there are exactly two possible leading order behaviors, i.e. two such singularity types :

$$\begin{aligned} &(1) \qquad x \sim A_1 \tau^{-1} \ , \ y \sim A_2 \tau^{-1} \ , \ z \sim A_3 \tau \\ &(2) \qquad x \sim B_1 \tau^{-1} \ , \ y \sim B_2 \tau \ , \ z \sim B_3 \tau^{-2} \qquad (\tau \equiv t - t_o) \end{aligned} \tag{3.2}$$

Examining first singularity type (1) we discover that Laurent series expansions can be developed about it for all values of γ,δ :

$$x = -\frac{1}{2\tau} + (-\frac{\gamma}{4} + i\frac{\delta}{4}) + a_1 \tau + \ldots$$

$$y = \frac{i}{2\tau} + (-\frac{\delta}{4} + i\frac{\gamma}{4}) + b_1 \tau + \ldots$$

$$z = A_3 \tau - 2A_3(1-\frac{\gamma}{4} + i\frac{\delta}{4}) \tau^2 + \ldots$$

where t_o , A_3 and a_1 (or b_1) are the 3 free constants. Thus the Painlevé conditions are fulfilled near this singularity.

Turning to the more interesting singularity type (2) we first make a convenient change of variables

$$y' \equiv y - \frac{\delta}{2} \tag{3.3}$$

and rewrite eq. (3.1) in the form

$$\begin{aligned} \dot{x} &= \gamma x - \delta y + z - 2y^2 \\ \dot{y} &= \gamma y + \frac{\gamma\delta}{2} + 2xy \\ \dot{z} &= -2z - 2zx \end{aligned} \tag{3.4}$$

where, for simplicity, we have dropped the primes on y . Developping the Laurent series expansion in this case one obtains

$$x = \frac{1}{\tau} - (\frac{\gamma}{2}+1) + \ldots \ , \ y = \frac{-\gamma\delta}{2} \tau + b_2 \tau^2 + \ldots \ , \ z = \frac{-1}{\tau^2} - \frac{\gamma}{\tau} + C_o + \ldots \tag{3.5}$$

where b_2 is a (second) free constant provided the following compatibility condition holds :

$$\gamma\delta = 0 \tag{3.6}$$

(a) $\gamma = 0$, $\delta \neq 0$:

Here the asymptotic expansions are as in (3.5), with C_o the third free constant to be determined by the initial conditions. Thus this case possesses the Painlevé property and is completely integrable in the sense of section I. One integral is easily found :

$$zy = C\, e^{-2t} \tag{3.7}$$

with the aid of which the first eq. (3.4) yields (using the second equation)

$$\ddot{y} = \frac{\dot{y}^2}{y} - 4y^3 + 2C\, e^{-2t} - 2\delta y^2 \ , \tag{3.8}$$

where C is an arbitrary constant. Remarkably enough with the substitutions :

$$Y \equiv e^t y \quad , \quad T \equiv e^{-t} \tag{3.9}$$

eq. (3.8) reduces to eq. XIII of Ince [8] which is the third of the six famous transcendental Painlevé equations ; hence complete integrability is verified.

(b) $\delta = 0$, $\gamma \neq 0$:

In this case the asymptotic expansions (3.5) possess a third free constant C_o required by the Painlevé property iff :

$$\gamma(\gamma+1) = 0 \tag{3.10}$$

Since $\gamma=0$ was already considered above we concentrate now on the $\gamma=-1$ case : First, we note that for $\gamma=-1$ there exists one integral

$$x^2 + (y+\frac{\delta}{2})^2 + z = De^{-2t} \ , \tag{3.11}$$

valid for all δ ! In the special case where $\delta=0$, a second integral is obtained by combining the second and third ODE's in (3.1)

$$zy = C\, e^{-3t} , \tag{3.12}$$

where C,D are arbitrary constants. With the aid of (3.11), (3.12) and the change of variables (3.9) one is led after some algebra to a single quadrature

$$\left(\frac{dY}{dT}\right)^2 = 4Y^2D - 4Y^4 - 4CY$$

which can be completed in terms of elliptic functions [27]. This ends the discussion of the completely integrable cases of (3.1) identified by the Painlevé property.

We would like to diwell, however, for a moment on a partially integrable case identified by the above analysis : It is the case $\gamma=-1$ and δ arbitrary for which only one integral has been discovered, i.e. (3.11). The violation of the compatibility condition at the first higher order forces us to introduce logarithmic terms in the asymptotic expansions near singularity type (2). The first few terms of these expansions are :

$$\begin{aligned} x &= \frac{1}{\tau} - \frac{1}{2} + a_1\tau + \ldots \\ y &= \frac{\delta}{2}\tau - \delta\tau^2 \ln \tau + b_2 \tau^2 + \ldots \\ z &= -\frac{1}{\tau^2} + \frac{1}{\tau} + C_o + \ldots \end{aligned} \tag{3.13}$$

with t_o , b_2 and C_o (or a_1) the 3 free constants and higher powers of $\tau(\ln \tau)$ entering in higher order terms of (3.13).

This is one more example of partial integrability of the type we encountered in the Lorenz equations of section I for $b = 2\sigma$ or $b = 1$, $\rho = 0$, cf. (1.4), (1.5). One might well ask whether many other such partially integrable systems exist, whose leading order behavior near a (movable) singularity is pole-like and only $\ell n\, \tau$ terms are needed to complete the asymptotic expansion at higher orders. Although we still have no theorem in this direction, we will show in the remainder of this paper that many such examples actually do exist. The importance of partial integrability is quite clear : For example, the integral (3.11) for (3.1) with $\gamma = -1$ (and any δ) means that as $t \to \infty$ all motion tends to a simple two-dimensional surface, which is a paraboloid about the $z < 0$ axis. For other completely integrable cases of more general multiple wave interactions, recently identified by the Painlevé property, the reader is referred to [28,29].

B. The Rikitake Dynamo Model

The Rikitake Two Disk Dynamo model, proposed for the description of the time variation of the earth's magnetic field [30] is

$$\begin{aligned} \dot{x} &= -\mu x + \beta y + yz \\ \dot{y} &= -\mu y - \beta x + xz \\ \dot{z} &= -xy + \alpha \end{aligned} \tag{3.14}$$

The analysis of (3.14) is particularly simple since it has only one type of singularity, near which the leading order behavior of its solutions is

$$x \sim i/\tau \ , \ y \sim -i/\tau \ , \ z \sim 1/\tau \qquad (\tau \equiv t - t_o) \tag{3.15}$$

Developing the asymptotic series to higher orders one easily finds that two more free constants are expected simultaneously at the second higher order [31]. The compatibility conditions for these free constants to enter with only integer powers of τ yield.

$$\begin{aligned} &\text{(a)} \qquad \alpha = 0 \quad \underline{\text{and}} \\ &\text{(b)} \qquad \underline{\text{either}} \ \ \beta = 0 \ , \ \underline{\text{or}} \ \ \mu = 0 \end{aligned} \tag{3.16}$$

The two cases possessing the Painlevé property can be easily integrated : with $\alpha = 0$ $\underline{\text{and}}$ $\beta = 0$ there is one integral

$$x^2 - y^2 = C^2 \, e^{-2\mu t} \tag{3.17}$$

and after the variable transformation

$$x+y \equiv C \, u \, e^{-\mu t} \quad , \quad T \equiv e^{-\mu t} \tag{3.18}$$

the Rikitake system leads to a single ODE of second order

$$\frac{d^2u}{dT^2} = \frac{1}{u} \left(\frac{du}{dT}\right)^2 - \frac{1}{T} \frac{du}{dT} - \frac{C^2}{4\mu^2} \left(u^3 - \frac{1}{u}\right) \tag{3.19}$$

which is again a special case of the 3d Painlevé transcendental equation [8]. On the other hand, the case $\alpha=0$ $\underline{\text{and}}$ $\mu=0$ is even simpler : Multiplying the first and second eq. (3.14) by $\dot{x}$ and $\dot{y}$ respectively, adding and subtracting yields the two integrals :

$$x^2 + y^2 + 2z^2 = C \tag{3.20a}$$

$$x^2 - y^2 + 4\beta z = D \ . \tag{3.20b}$$

Using (3.20) the complete integration of the system can be performed in terms of elliptic functions.

It is interesting now to ask whether the partial fulfillment of conditions (3.16) yields partially integrable models, as was the case with $\gamma=-1$ (and any δ) in the 3-wave interaction system of section 3A. The answer is that indeed it does :

Take, for example, the case $\beta=0$ (with $\alpha\mu \neq 0$). The integral (3.17) still exists and the motion as $t \to \pm\infty$ (for $\mu \gtrless 0$) is attracted by any of the four limit cycles

$$x_1 = \varepsilon_1 y$$

$$(z-\varepsilon_1 \mu)^2 + y^2 = 2\varepsilon_2 \alpha \ln |y| \quad (\varepsilon_i = \pm 1)$$

On the other hand, if $\mu=0$, one integral also exists for all α and β :

$$x^2 - y^2 + 4\beta(z-\alpha t) = D \tag{3.21}$$

but no further integration appears possible.

But the pole-like behavior at leading order accompanied by only $\ln \tau$ terms at higher orders may yield more than partial integrability. For example if both μ and β vanish, the Rikitake model possesses two integrals and hence is completely integrable : The first integral is (3.21) (with $\beta=0$). A second integral can be derived by noting first that the operations for deriving the integral (3.20a) here give

$$\frac{d}{dt}(x^2+y^2+2z^2) = 4z\alpha \tag{3.22}$$

Now the $\dot{x}$ and $\dot{y}$ equations in (3.14) with $\beta=0$ and $\mu=0$ immediately give $z = (\dot{x}+\dot{y})/(x+y)$, which, when substituted in (3.22) yields the second integral

$$x^2+y^2+2z^2 - 4\alpha \ln |x+y| = E \ . \tag{3.23}$$

We observe, therefore, at least on these models that it is often possible to identify partially integrable cases, with only one integral [e.g. (3.17)] by partially satisfying the conditions of the Painlevé property [e.g. $\beta=0$ and any α,μ]. This does not seem to hold always : For example, for $\alpha=0$ and any β,μ, cf. (3.16), no simple integral of the form (3.20)-(3.23) appears to exist. Still, we have discovered so many partially integrable examples using the above approach (see also Section V) that we believe this subject warrants further investigation.

In the next section, we begin such an investigation by showing for a specific Nth order model a direct relation between the partial fulfillment of the Painlevé conditions and the existence of N-2 integrals of the motion.

IV - PARTIAL INTEGRABILITY OF A GENERALIZED ROSSLER SYSTEM

In this section, we demonstrate on a simple model of N first order ODE's that the existence of integrals can be directly connected to the fulfillment of the Painlevé condition of pole-like behavior at lowest order. Our model is an Nth order generalization of a system due to Rössler

$$\dot{x}_1 = \sum_{j=1}^{N} x_j$$

$$\dot{x}_k = \sum_{j=1}^{N} a_{kj}\, x_j \qquad k=2,\ldots,N-1 \qquad (4.1)$$

$$\dot{x}_N = 1 + x_1\, x_N$$

which for $N=3$ has been shown to exhibit wildly chaotic behavior for some parameter values [3]. This is perhaps the simplest, non-trivial system : $N-1$ of its equations are linear but the nonlinear term in its Nth equation is sufficient to render it, in general, non-integrable.

We are seeking here conditions on the a_{kj} parameters of (4.1), for which we can find as many integrals of motion as possible. First of all, owing to the N-1 linear equations of (4.1), we may even guess the form of these integrals

$$e^{-\lambda_k t} \sum_{j=1}^{N-1} b_{kj}\, x_j = C_k \qquad k=1,2,\ldots,N-1 \qquad (4.2)$$

The C_k in (4.2) are arbitrary constants and the b_{kj} are to be determined by substituting (4.2) in the first N-1 eqs. of motion (4.1). This readily yields the eigenvalue equation

$$\underset{\approx}{A} \cdot \underset{\sim}{b}_k = \lambda_k \underset{\sim}{b}_k \tag{4.3}$$

$k=1,\ldots,N-1$, where

$$\underset{\sim}{b}_k \equiv (b_{k,1},\ldots,b_{k,N-1})^T \tag{4.3a}$$

(T denoting transpose) and $\underset{\approx}{A}$ is the $(N-1) \times (N-1)$ matrix

$$\underset{\approx}{A} \equiv \begin{pmatrix} 1 & a_{2,1} & a_{3,1} & \cdots & a_{N-1,1} \\ 1 & a_{2,2} & a_{3,2} & \cdots & a_{N-1,2} \\ 1 & a_{2,3} & a_{3,3} & & \\ \vdots & \vdots & \vdots & & \\ 1 & a_{2,N-1} & & & a_{N-1,N-1} \end{pmatrix} \tag{4.3b}$$

At the same time we must require that the coefficients of the x_n terms resulting from the above substitution vanish. This yields the following equations for the $a_{j,N}$'s :

$$b_{k,1} + \sum_{j=2}^{N-1} b_{k,j}\, a_{j,N} = 0 \qquad k=1,\ldots,N-1 \tag{4.4}$$

Now, in general, given any coefficient matrix $\underset{\approx}{A}$, cf. (4.3b) eq. (4.3) can be solved for its $N-1$ eigenvectors $\underset{\sim}{b}_k$ and eigenvalues λ_k. Thus we may think of (4.4) as $N-1$ necessary and sufficient conditions for the $a_{j,N}$'s, so that $N-1$ integrals (4.2) exist. However since there are only $N-2$ $a_{j,N}$'s they can only satisfy, in general, $N-2$ of the equations (4.4) and hence only $N-2$ integrals of the form (4.2) are to be expected.

Suppose now we wanted to determine for which values of the parameters $a_{k,j}$ the solutions of (4.1) possess the Painlevé property. First of all, it is not difficult to check that system (4.1) has only one singularity type near which the leading order behavior of the x_k's is

$$x_k \sim C_k \tau^{-1} \quad , \quad k = 1,2,\ldots,N-1$$

$$x_N \sim C_N \tau^{-2} \qquad (\tau \equiv t-t_o) \quad . \tag{4.5}$$

Inserting (4.5) in (4.1) one immediately finds

$$C_1 = -2 = -C_N \; ; \; C_k = -2a_{k,N} \; , \qquad k=2,\ldots,N-1 \tag{4.6}$$

In order to determine at which order free constants are to arise, one substitutes in (4.1)

$$x_k = C_k \tau^{-1} + \alpha_k \tau^{-1+r} \quad , \; k=1,\ldots,N-1$$

$$x_N = C_N \tau^{-2} + \alpha_N \tau^{-2+r}$$

and derives, as usual, linear equations for the α_k's whose coefficient matrix turns out to have vanishing determinant iff

$$(r+1)(r-1)^{N-2}(r-2) = 0 \tag{4.7}$$

This implies that N-2 free constants are expected at the first higher order and one free constant at second higher order. One necessary condition, however, for this to happen is that all τ^{-1} terms on the rhs of the first N-1 eq. (4.1) cancel out, i.e.

$$
\begin{aligned}
0 &= \sum_{j=1}^{N-1} C_j + \beta \\
0 &= \sum_{j=1}^{N-1} a_{kj}\, C_j + a_{k,N}\, \beta \,, \qquad k=2,\ldots,N-1
\end{aligned}
\tag{4.8}
$$

where β is the coefficient of the τ^{-1} term of x_N.

Using (4.6) and defining

$$
\underset{\sim}{a} \equiv (1 \,,\, a_{2,N} \,,\, \cdots \,,\, a_{N-1,N})^T \tag{4.9}
$$

we rewrite (4.8) as follows :

$$
\underset{\sim}{0} = (\underset{\approx}{A}^T - \frac{\beta}{2}\, \underset{\approx}{I}) \cdot \underset{\sim}{a} \tag{4.10}
$$

Now form the matrix

$$
\underset{\approx}{B} \equiv (b_{kj}) \qquad k,j = 1,\ldots,N-1 \tag{4.11}
$$

and multiply both sides of (4.10) from the left by $\underset{\approx}{B}$ to obtain, with the aid of (4.3)

$$
\underset{\sim}{0} = (\underset{\approx}{\Lambda} - \frac{\beta}{2}\, \underset{\approx}{I}) \cdot \underset{\approx}{B}.\underset{\sim}{a} \tag{4.12}
$$

where $\underset{\approx}{\Lambda} \equiv (\Lambda_{kj})$ is the diagonal matrix with elements

$$
\Lambda_{kj} = \lambda_k\, \delta_{k,j} \qquad , \qquad k,j = 1,\ldots,N-1
$$

Thus, remarkably enough the condition (4.12) for the solutions of (4.1) to have pole-like behavior near the (movable) singularity $t = t_o$, i.e.

$$
\underset{\approx}{B}.\underset{\sim}{a} = \underset{\sim}{0}
$$

is identical to the condition (4.4) for the existence of N-2 integrals (4.2), cf. (4.9), (4.11).

More examples of such connections between partial integrability and the Painlevé conditions of singularity analysis are given in the next section. However, to what extent these results can be generalized is not known, at present. It seems difficult even to formulate a statement which one would attempt to prove rigorously. One way to proceed, of course, is to analyse specific classes of systems like the one of this section, for which the connection between singularities and integrability can be explicitly demonstrated.

V - THE RABINOVICH SYSTEM

In this section we treat one more physical example with the methods of singularity analysis. In particular, we find completely and partially integrable cases of a three wave interaction system due to Rabinovich [2]

$$\begin{aligned} \dot{x} &= hy - \nu_1 x + yz \\ \dot{y} &= hx - \nu_2 y - xz \\ \dot{z} &= -\nu_3 z + xy \end{aligned} \tag{5.1}$$

where ν_1, ν_2, ν_3 are the damping rates and h is proportional to the driving amplitude of the "feeder" wave. Our main purpose here (as in sections III and IV) is to demonstrate that by partially satisfying the conditions of complete integrability one can integrate (5.1) once and show that all solutions approach (as $t \to \infty$) simple two-dimensional surfaces, on which all motion is regular and predictable.

Clearly, the only singularity type of (5.1) has the leading order behavior

$$x \sim -i/\tau \quad , \quad y \sim 1/\tau \quad , \quad z \sim i/\tau \qquad (\tau \equiv t-t_o) \tag{5.2}$$

like the Rititake model, cf. eq. (3.15). Inserting now Laurent power series in (5.1) beginning with (5.2) one finds that the two remaining free constants (the first one is t_o) are expected to enter simultaneously at the second higher order, cf. section III.A. This will happen with integer powers of τ (and no $\ell n\, \tau$ terms) provided certain compatibility conditions are satisfied. Thus, one finds that (5.1) has the Painlevé property and hence is completely integrable in the following cases :

(1) $h \neq 0$ (scaling $h=1$) and

either (a) $\nu_1 - \nu_2 = 2i \quad , \quad \nu_3 = 0$

or (b) $\nu_1 + \nu_2 = \nu_3 \quad , \quad \nu_1 - \nu_2 = 2i\nu_1\nu_2$

(2) $h = 0$, and

either (a) $\nu_2 - \nu_1 = \nu_3$

or (b) $\nu_j = \nu_k \ , \ \nu_\ell = 0 \qquad (j,k,\ell \text{ unequal})$

Also the case $\nu_1 = \nu_2 = \nu_3 = 0$ is completely integrable (for all h) in terms of elliptic functions.

Take, for example, case (1a) above : Multiplying the second equation (5.1) by i and adding to the first one we get

$$\frac{\dot{u}}{u} = i\text{-}\nu_1 - iz \tag{5.3}$$

where

$$u \equiv x+iy \equiv \rho e^{i\theta} \tag{5.4}$$

Differentiate now (5.3) once and use (5.4) and the third eq. (5.1) to obtain, upon separating real and imaginary parts :

$$\rho = C\, e^t \quad , \quad \ddot{\phi} = C^2\, e^{2t} \sin\phi \qquad (\phi \equiv 2\theta) \tag{5.5}$$

where C is an arbitrary constant. Remarkably enough with the change of variables.

$$u = e^{i\phi} \qquad T = \frac{C^2}{8}\, e^{2t} \ , \tag{5.6}$$

the ϕ-equation of (5.5) can be transformed to the third Painlevé transcendental equation [8]. In an entirely similar way, using (5.4) etc., one can integrate case (2b) above, with $\nu_1 = \nu_2$, say, and $\nu_3 = 0$.

Turning now to the partially integrable cases we obtain the following results :

(i) $h \neq 0$, $\nu_1=\nu_2=\nu$, $\nu_3 = 2\nu$, suggested by the completely integrable case (1b), possesses the integral

$$x^2+y^2-4z = C\, e^{-2\nu t} , \qquad \nu > 0 ,$$

whence all solutions tend (as $t \to \infty$) to a paraboloid of revolution around the $z>0$ axis.

(ii) $h \neq 0$, $\nu_1=\nu_2=\nu_3=\nu$, suggested by the corresponding completely integrable $\nu=0$ case. The integral here is

$$x^2-y^2-2z^2 = C\, e^{-2\nu t} , \qquad \nu > 0 ,$$

and all motion "collapses", as $t \to \infty$, on a double ellipsoidal cone about the x-axis with apex at the origin

(iii) $h = 0$, $\nu_1=\nu_2=\nu$, suggested by the completely integrable (2b) above, has one integral

$$x^2+y^2 = C\, e^{-2\nu t} \qquad \nu > 0 .$$

Since there is no driving wave here, the motion is globally damped and all solutions are attracted to the origin as $t \to \infty$.

All the above partially integrable cases of the Rabinovich system have the same analytic structure as the ones of sections III and IV : their leading order behavior near a (movable) singularity at $t=t_o$ is a pole, with $\ell n(t-t_o)$ terms [and integer powers of $(t-t_o)$] needed at higher orders to capture the general asymptotic expansions of the solutions.

VI - CONCLUDING REMARKS

In this paper, we have attempted to demonstrate the usefulness of the singularity analysis in integrating the equations of motion of dynamical systems. We concentrated, in particular, on several non-Hamiltonian systems of physical interest, complementing thus similar earlier work on Hamiltonian systems.

We defined complete integrability for non-Hamiltonian systems and used the Painlevé property to identify many new, completely integrable third order models. While subsequently integrating these models, we discovered several partially integrable cases by partially satisfying the conditions for the Painlevé property. These cases are distinguished by the fact that their solutions near every movable singularity (at $t=t_o$) in the complex time plane start with a pole and only admit $\ell n(t-t_o)$ terms at higher orders.

Thus we have used singularity analysis to identify not only completely integrable dynamical systems, but also partially integrable systems of n first order ODE's whose chaotic behavior, if any, is restricted to $k(< n)$-dimensional spaces. One must be very careful, however, when associating integrability with different types of singularities ; there are many examples of completely (and partially) integrable systems, whose analytic structure is much more complicated than poles, or poles plus logarithms. Clearly these examples remain at present beyond the reach of the Painlevé analysis.

The Painlevé property (i.e. that all movable singularities be poles) and its relaxation (i.e. that these singularities be poles at leading order with logarithmic terms at higher orders), have

respectively yielded a wealth of new, completely integrable and "weakly chaotic" dynamical systems. However, these results only represent the first step in a direction of research whose ultimate goal is a classification of the chaotic properties of dynamical systems according to the analytic structure of their solutions in the complex time plane.

ACKNOWLEDGEMENTS

One of us (T. Bountis) gratefully acknowledges the support of the P.O.E. grant DE-AC02-83ER13052 and thanks the "Nederlandse Organisatie voor Zuiver Wetenschapplijk Onderzoek" (ZWO), under whose auspices this work was carried out as part of the research programme of "Stichting voor Fundamenteel Onderzoek der Materie" (FOM).

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CHAPITRE 7

SYSTÈMES HAMILTONIENS DÉPENDANT DU TEMPS

Dans cette partie, nous nous intéressons à des systèmes régis par un hamiltonien $H(x,\dot{x},t)$ qui dépend du temps de manière explicite. L'étude de ces systèmes est beaucoup plus délicate que celle des systèmes autonomes puisque l'hamiltonien n'est plus une constante du mouvement.

Un exemple d'importance historique est celui de l'oscillateur harmonique à fréquence variable :

$$H = \frac{1}{2}\left[\dot{x}^2 + \omega^2(t)x^2\right] \qquad (1)$$

$\omega(t)$ est une fonction arbitraire du temps. Cet hamiltonien sert à décrire le mouvement d'une particule chargée, non relativiste, dans un champ électromagnétique.

Ce système possède comme invariant :

$$I = \frac{1}{2}\left[(\rho\dot{x} - x\dot{\rho})^2 + \left(\frac{x}{\rho}\right)^2\right] \qquad (2)$$

pourvu que ρ satisfasse à l'équation différentielle suivante :

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3} \qquad (3)$$

Cet invariant a d'abord été trouvé par Ermakov [1] en éliminant $\omega^2(t)$ entre l'équation *(3)* et l'équation du mouvement pour x.

Environ cent ans plus tard, Lewis [2] a retrouvé ce résultat en appliquant la théorie asymptotique de Kruskal.

Lewis et Riesenfeld [3] ont utilisé cet invariant *(2)* pour résoudre de manière exacte le problème de l'oscillateur quantique dépendant du temps.

D'ailleurs, ce qui motive le plus la recherche d'invariants dépendant du temps, pour de tels systèmes, c'est le fait que, pour le problème quantique, l'invariant peut jouer le rôle que l'hamiltonien joue pour les systèmes indépendants du temps.

Ainsi, dans le cas cillateur, on peut définir à partir des quantités x et ρ deux op a et a^+ par :

$$a = \frac{1}{(2\hbar)^{1/2}} \left[\frac{x}{\rho} + i\rho\dot{x}\right] \qquad (4)$$

$$a^+ = \frac{1}{(2\hbar)^{1/2}} \left[\frac{x}{\rho} - i\rho\dot{x}\right]$$

où $[x,p] = i\hbar$.

On peut alors mo e I vérifie :

$$\frac{dI}{dt} = \frac{1}{i\hbar} [I,H] \qquad (5)$$

et

$$I = \hbar\left[a^+a + \frac{1}{2}\right] \qquad (6)$$

On déduit de l'(5) que les valeurs propres de I sont constantes et on calcule à de (6) le spectre de I. Il est ainsi possible de trouver une sol xacte pour l'oscillateur dépendant du temps en terme des fonct opres de I.

La méthode d'Er eut aussi être utilisée pour étudier le problème suivant qui se e comme une généralisation du cas de l'oscillateur harmoniqu

Considérons le ions :

$$\ddot{x} + \omega^2(t)x = \frac{\rho}{x^3} \qquad (7)$$

$$\ddot{\rho} + \omega^2(t)\rho = \frac{x}{\rho^3} \qquad (8)$$

ω, f et g son onctions arbitraires.
x et ρ dépend licitement du temps.

Ray et Reid [montré que la quantité :

$$I = \frac{1}{2}\left[(\rho\dot{x} + \varphi\left(\frac{x}{\rho}\right) + \theta\left(\frac{\rho}{x}\right)\right] \qquad (9)$$

où :

$$\varphi\left(\frac{x}{\rho}\right) = 2\int^{x}) \, du$$

$$\theta\left(\frac{\rho}{x}\right) = 2 \, (u) \, du$$

vérifie $\frac{dI}{dt}$ i x et ρ vérifient (7) et (8).

I est donc un invariant dépendant du temps.

Comme on le voit immédiatement, l'inconvénient de cette approche c'est que l'on doit connaître non seulement l'équation de mouvement *(7)* du système, mais aussi l'équation auxiliaire *(8)* pour déterminer l'invariant *(9)*. Or, en face d'une équation donnée, on ne connaît en général pas l'équation auxiliaire. C'est pour cela que le besoin de méthodes différentes se fait sentir.

Au moins quatre méthodes différentes ont été utilisées avec succès : la méthode de Kruskal des invariants adiabatiques [5], la méthode de Leach des transformations canoniques dépendant du temps [6], la méthode développée par Lutzky basée sur le théorème de Noether [7], et, enfin, une méthode de recherche directe d'invariants proposée par Lewis et Leach [8].

Dans la première des deux publications que nous allons présenter, nous appliquons la méthode de Lewis-Leach à une famille d'hamiltoniens proposée par Sarlet.

Celui-ci a cherché à déterminer tous les hamiltoniens dépendants du temps, à un degré de liberté, pour lesquels la méthode de Kruskal donne des invariants exacts. A titre d'exemple, il s'est intéressé à une classe d'hamiltoniens polynomiaux de la forme :

$$H = \frac{1}{2} a(t)x^k\dot{x}^2 + b(t)x^l\dot{x} + c(t)x^m \qquad (10)$$

Il a ainsi pu déterminer certaines valeurs des paramètres k, l, m pour lesquelles un invariant existe.

L'application de la méthode directe nous a permis de calculer des invariants exacts linéaires ou quadratiques dans les impulsions pour une famille de paramètres plus étendue que celle initialement trouvée par Sarlet.

La seconde publication traite des systèmes à deux dimensions. Pour de tels systèmes, fort peu de résultats existent jusqu'à présent et notre travail peut se présenter comme une généralisation à deux dimensions, des résultats de Lewis et Leach.

Nous avons calculé une constante du mouvement en généralisant de manière naturelle la méthode de recherche directe implémentée dans le cas autonome (voir chapitre 5).

Nous nous limitons à chercher les systèmes régis par un hamiltonien de la forme :

$$H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + V(x,y,t)$$

qui admettent des invariants linéaires ou quadratiques dans les vitesses, c'est-à-dire de la forme :

$$C = g_0\dot{x} + g_1\dot{y} + h$$

ou

$$C = f_0\dot{x}^2 + f_1\dot{x}\dot{y} + f_2\dot{y}^2 + g_0\dot{x} + g_1\dot{y} + h$$

L'hamiltonien H n'étant pas invariant par renversement du temps, la constante C ne l'est pas non plus et donc toutes les parités dans les puissances des vitesses sont effectivement présentes. Comme dans le cas autonome, la condition $\frac{dC}{dt} = 0$ va imposer des contraintes aux fonctions f_i, g_i et h. La forme des f_i est la même que dans le cas autonome. Par contre, pour une constante quadratique, on vérifie que l'existence des fonctions g_i modifie l'équation de compatibilité pour le calcul de h. Un terme inhomogène est en effet rajouté à cette équation aux dérivées partielles pour V. Néanmoins, on peut toujours utiliser pour l'étudier la même classification que dans le cas homogène. Une fois que V a été déterminé, il reste une dernière équation (en général, non linéaire pour V) à satisfaire. Cette relation limite sévèrement les formes de potentiels admissibles. Malgré cela, nous avons été capables de déterminer des familles de potentiels intégrables ainsi que les invariants correspondants.

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PHYSIQUE THÉORIQUE. – *Une classe d'Hamiltoniens dépendant du temps qui possèdent un invariant exact.* Note (*) de **Basile Grammaticos, Bernadette Dorizzi** et **Radovan Padjen,** présentée par Roger Balian.

Nous appliquons la théorie développée par Lewis et Leach à une famille d'Hamiltoniens, dépendant du temps, proposée par Sarlet. Nous obtenons ainsi des invariants exacts linéaires ou quadratiques dans l'impulsion, pour une classe d'Hamiltoniens plus large que celle obtenue initialement par Sarlet.

THEORETICAL PHYSICS. – A Class of Time-Dependent Hamiltonians Which Possess an Exact Invariant.

We apply the theory developed by Lewis and Leach on a family of time-dependent Hamiltonians proposed by Sarlet. We obtain thus exact invariants, linear or quadratic in the momentum, for a class of Hamiltonians larger than the one initially obtained by Sarlet.

La modélisation de phénomènes physiques non stationnaires conduit très souvent à des Hamiltoniens qui dépendent explicitement du temps. En plus, l'étude des systèmes quantiques, dont l'homologue classique présente un comportement chaotique procède souvent par l'intermédiaire des systèmes hamiltoniens à une dimension dépendant explicitement du temps ([1], [2]). La dépendance du temps a comme conséquence directe la non existence de l'intégrale de l'énergie en tant que constante de mouvement. Des moyens considérables ont donc été déployés afin de pouvoir identifier ceux des systèmes hamiltoniens dépendant du temps qui possèderaient un invariant exact. L'utilité d'un tel invariant est capitale. Il permet en général une transcription du problème qui amène des simplifications importantes. Ceci est d'autant plus vrai dans le cas des systèmes quantiques où la solution de l'équation de Schrödinger, équation aux dérivées partielles, peut-être ramenée à la résolution d'une équation différentielle ordinaire ([3], [4]). De plus, dans le cas où l'invariant est quadratique dans l'impulsion, une transformation canonique permet de réécrire ce système sous forme autonome, c'est-à-dire avec un Hamiltonien indépendant du temps.

Dans un travail récent, Lewis et Leach [5] ont examiné des Hamiltoniens de la forme :

$$\text{(1)} \qquad H = \frac{1}{2} P^2 + V(x, t)$$

et obtenu la forme générale de V qui assure l'existence d'un invariant linéaire ou quadratique dans l'impulsion p. Un problème analogue avait été examiné par Sarlet [6] pour des Hamiltoniens de la forme :

$$\text{(2)} \qquad H = \frac{1}{2} a(t) x^k P^2 + b(t) x^l P + c(t) x^m,$$

pour lesquels il a obtenu une linéarisation en p utilisant la méthode de Kruskal [7], et construisant par la suite explicitement un invariant linéaire ou quadratique en p, pour certaines valeurs de k, l, m. Dans le présent travail, nous allons appliquer la méthode de Lewis-Leach aux Hamiltoniens de Sarlet et montrer qu'il est possible d'obtenir un invariant exact pour un ensemble plus grand d'Hamiltoniens que celui donné initialement par Sarlet.

Nous commençons par une transformation canonique sur P, x qui permet d'écrire l'Hamiltonien de Sarlet sous une forme plus simple, se rapprochant de (1). On introduit :

$$\text{(3)} \qquad q = \int \frac{dx}{\sqrt{a(t)}\, x^k} = \frac{1}{\sqrt{a}} \frac{x^{1-k/2}}{1-k/2}, \qquad p = \sqrt{a}\, x^{k/2} P + \frac{b x^l}{\sqrt{a}\, x^{k/2}},$$

avec les précautions habituelles $a \neq 0$, $k \neq 2$. On peut vérifier aisément que $\{p, q\} = 1$.

L'Hamiltonien (2) s'écrit :

$$H = \frac{1}{2} p^2 + c\left[\left(1 - \frac{k}{2}\sqrt{a}\right)q\right]^{m/(1-k/2)} - \frac{b^2}{2a}\left[\left(1 - \frac{k}{2}\sqrt{a}\right)q\right]^{(2l-k)/(1-k/2)} \tag{4}$$

Le cas $k = 2$ conduit à $q = 1/\sqrt{a} \ln x$ et à un potentiel :

$$V = ce^{a\sqrt{a}m} - \frac{b^2}{2a} e^{a\sqrt{a}2(l-1)}. \tag{5}$$

De telles formes de potentiels ne faisant pas partie de la famille donnée par Lewis-Leach, il n'existe qu'une seule solution : $m = 0$, $l = 1$, auquel cas le potentiel est indépendant de la position et l'invariant est tout simplement $C = p$, en accord avec les résultats de Sarlet. En résumant les résultats de Lewis-Leach, un invariant du type $C = fp + g$ existe, si et seulement si le potentiel est de la forme :

$$V_1 = \tau + \frac{\dot{\sigma}}{\rho} q - \frac{\ddot{\rho}}{2\rho} q^2, \tag{6}$$

avec τ, σ, ρ fonctions du temps.

La forme exacte de l'invariant dans ce cas est :

$$C_1 = \rho p + \sigma - \dot{\rho} q. \tag{7}$$

Pour le cas des invariants quadratiques nous avons :

$$V_2 = \tau + \frac{\dot{\sigma}}{\rho} q - \frac{1}{2}\frac{\ddot{\rho}}{\rho} q^2 + \frac{1}{\rho^2} G\left(\frac{q - \upsilon}{\rho}\right), \tag{8}$$

où :

$$\upsilon = -\rho \int_0^t \frac{\sigma(t')}{\rho^2(t')}\, dt'$$

et G est une fonction quelconque de son argument. L'invariant devient dans ce cas :

$$C_2 = \frac{1}{2}\left[\rho(p - \dot{\upsilon}) - \dot{\rho}(q - \upsilon)\right]^2 + G\left(\frac{q - \upsilon}{\rho}\right). \tag{9}$$

Donc un invariant linéaire existe seulement pour des potentiels harmoniques, et un invariant quadratique seulement pour les systèmes d'Ermakov.

En ce qui concerne l'existence des invariants linéaires, l'Hamiltonien de Sarlet peut être ramené au cas harmonique seulement lorsque $2m/(2-k)$ et $(4l-2k)/(2-k)$ sont les entiers 0, 1 ou 2. Il existe donc en tout neuf cas d'Hamiltoniens de Sarlet possèdant un invariant linéaire. Dans son analyse Sarlet avait obtenu comme seules solutions les cas $l = 1$, $m = 1 - (k/2)$ et $2l = k$, $m = 1 - (k/2)$ qui font partie, évidemment, des neuf cas cités plus haut.

Dans le cas de l'invariant quadratique, la situation est un peu plus compliquée à cause de la présence de la fonction G dans le potentiel. Cependant comme le potentiel de Sarlet a la forme :

$$V_S = \varphi(t) q^{\nu} + \chi(t) q^{\mu}, \tag{10}$$

le choix des formes pour G se trouve sévèrement limité. Les choix $G(\xi) = 1/2\, \varkappa \xi^n$ avec $n = 0, 1, 2$ donnent pour V_2 des formes qui sont les mêmes que celles pour lesquelles un invariant linéaire existe. Mais ceci ne veut pas dire que l'invariant soit linéaire. Il est facile de le comprendre dans le cas de l'oscillateur harmonique. En effet si :

$$V = \frac{1}{2}\omega^2(t) q^2, \tag{11}$$

nous avons comme conditions pour l'existence d'un invariant linéaire :

$$-\frac{1}{2}\frac{\ddot{\rho}}{\rho}=\frac{1}{2}\omega^2(t),$$

donc :

$$\ddot{\rho}+\omega^2(t)\,\rho=0. \tag{12}$$

Dans le cas de l'invariant quadratique ($G=1/2\,\varkappa\xi^2$), on trouve :

$$\frac{1}{2}\frac{\varkappa}{\rho^4}-\frac{1}{2}\frac{\ddot{\rho}}{\rho}=\frac{1}{2}\omega^2$$

et :

$$\ddot{\rho}+\omega^2(t)\,\rho-\frac{\varkappa}{\rho^3}=0. \tag{13}$$

La relation entre ces deux équations est, évidemment, bien connue [9]. Une façon intéressante de relier ces deux équations est de remarquer que l'équation (12) par une transformation simple $\psi=\rho^2$ peut se ramener à une des cinquante équations étudiées par Painlevé. En particulier Gambier [10] a obtenu la solution de l'équation (13) à l'aide d'une équation linéaire auxiliaire qui dans ce cas précis se réduit à (12).

Un des deux cas d'invariants quadratiques présentés par Sarlet $l=1, m=2-k$ correspond à ce choix de G, et on peut remarquer que ce cas est déjà présent dans notre analyse des invariants linéaires. Deux choix de G sont encore possibles correspondant à des puissances entières $G=1/2\,\varkappa\xi^n$, $n=3, 4$. Pour $n\geqq 5$ il n'est plus possible de ramener le potentiel (7) à la forme de Sarlet (9). Et même pour les cas $n=3, 4$ on obtient des contraintes sur les fonctions $\varphi(t)$, $\chi(t)$. Nous l'illustrons à l'aide d'un exemple basé sur l'oscillateur quartique :

$$V_S=\frac{1}{2}\frac{\varkappa}{\varphi^6(t)}\,q^4+\frac{1}{2}\omega^2(t)\,q^2. \tag{14}$$

Dans ce cas on trouve immédiatement $\rho=\varphi, \upsilon=0, \tau=0$, et l'équation de compatibilité s'écrit :

$$-\frac{1}{2}\frac{\ddot{\rho}}{\rho}=\frac{1}{2}\omega^2.$$

Donc un oscillateur quartique possède un invariant quadratique seulement lorsque ses deux fréquences obéissent à l'équation :

$$\ddot{\varphi}+\omega^2\varphi=0. \tag{15}$$

Dans le cas que nous venons d'analyser nous avons trouvé $\upsilon=0$. Ce cas présente un intérêt particulier parce qu'il conduit à des solutions pour le potentiel de Sarlet qui ne sont pas forcément des puissances entières. En effet pour $\upsilon=0$, V_2 s'écrit :

$$V_2=\tau-\frac{1}{2}\frac{\ddot{\rho}}{\rho}\,q^2+\frac{1}{\rho^2}\,G\left(\frac{q}{\rho}\right). \tag{16}$$

Pour identifier V_2 avec le potentiel (10) il faut :

Soit $\ddot{\rho}=0, \tau=0$; soit $\ddot{\rho}=0, \tau\neq 0$, soit $\tau=0, \ddot{\rho}\neq 0$ ce qui est le cas le plus intéressant. Nous avons dans ce cas en général $G=\lambda x^\nu+(1/2)\,\varkappa x^2$ et donc :

$$V=\frac{\lambda q^\nu}{\rho^{\nu+2}}+\left(\frac{\varkappa}{2\rho^2}-\frac{1}{2}\frac{\ddot{\rho}}{\rho}\right)q^2. \tag{17}$$

En réécrivant le potentiel de Sarlet comme :

$$V_S = \lambda \frac{q^\nu}{\varphi^{\nu+2}} + \frac{1}{2}\chi q^2, \tag{18}$$

nous obtenons $\rho = \varphi$ et :

$$\frac{1}{2}\chi = \frac{\varkappa}{2\rho^2} - \frac{1}{2}\frac{\ddot{\rho}}{\rho}.$$

Donc pour que le potentiel (23) possède un invariant quadratique il faut que φ et χ obéissent à l'équation :

$$\ddot{\varphi} + \chi\varphi - \frac{\varkappa}{\varphi^3} = 0. \tag{19}$$

Le cas de l'oscillateur quartique traité plus haut est un cas particulier de (21). Le cas $l = k-1, m = 2-k$ obtenu par Sarlet en fait partie aussi et il correspond à $\nu = -2$. Dans ce cas le terme $\lambda q^\nu/\rho^{\nu+2}$ devient indépendant du temps. En examinant la forme précise (4) de l'Hamiltonien de Sarlet nous obtenons :

$$\lambda = \frac{2b}{(2-k)^2 a^2}.$$

Ceci donc impose $b \propto a$ qui est exactement la condition obtenue par Sarlet dans son analyse.

Cependant notre solution (23) est plus générale et donne toute une famille d'Hamiltoniens qui possèdent un invariant quadratique à condition que les paramètres du potentiel obéissent à l'équation (24).

Ceci complète notre analyse de l'Hamiltonien de Sarlet. Nous avons réduit cet Hamiltonien à l'aide d'une transformation canonique à une forme qui pourrait être analysée à l'aide de la théorie de Lewis-Leach. L'application de cette dernière nous a permis d'identifier un grand nombre de cas où l'Hamiltonien de Sarlet possède un invariant exact. Nos résultats permettent ainsi d'étendre les résultats trouvés précédemment par Sarlet.

(*) Remise le 21 mars 1983.

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Département de Mathématiques (MIT), Centre National d'Études des Télécommunications,
38-40, *avenue du Général-Leclerc* 92131 *Issy-les-Moulineaux.*

TWO - DIMENSIONAL TIME - DEPENDENT HAMILTONIAN SYSTEMS WITH AN EXACT INVARIANT

B. GRAMMATICOS
B. DORIZZI

Dépt. des Mathématiques Appliquées (MTI). Centre National d'Etudes des des Télécommunications - 38-40 ave. du Général Leclerc -
92131 ISSY LES MOULINEAUX (FRANCE)

Abstract

We present a direct approach to investigate the existence of an exact invariant for two - dimensional Hamiltonians, in which the potential depends explicitly on time. The method is based on an expansion of the invariant in the velocities. The problem is solved completely for invariants linear and quadratic in the momenta. Our results contain as a particular case, the results of Lewis and Leach on one dimensional systems.

. A paraître dans Journal of Mathematical Physics.

I - INTRODUCTION

The theoretical description of non stationary physical phenomena often leads to time - dependent Hamiltonians. An example of historical importance is the description of the motion of a charged particle moving in an electromagnetic field. The Hamiltonian of the system can, in some cases at least, be reduced to the Hamiltonian of a harmonic oscillator, the frequency of which depends on time : $H = \frac{1}{2}(\dot{x}^2 + \omega^2(t)\, x^2)$. Lewis [1] has shown that an exact invariant, i.e. a conserved quantity, can be constructed for this problem :

$$C = \frac{1}{2}\left[\frac{x^2}{\rho^2} + (\rho\dot{x} - \dot{\rho}x)^2\right]$$ in terms of an auxiliary function $\rho(t)$ which is solution of the equation $\ddot{\rho} + \omega^2(t)\,\rho = \frac{1}{\rho^3}$. The derivation of the invariant can be traced back to Ermakov [2] who derived it in 1880. Gambier [3], in 1909, has also analysed the equation for ρ, or rather for $\Psi = \rho^2$, from the point of view of the Painlevé property. He has integrated it by reducing it to a linear equation, which is exactly the equation for the harmonic oscillator $\ddot{x} + \omega^2(t)\, x = 0$, and obtained the invariant in the course of his analysis. The importance of the result of Lewis stems from the fact that he used the invariant in order to construct the solution of the quantum time - dependent oscillator [4], thus reducing the solution of a P.D.E. (the Schrödinger equation) to the solution of an O.D.E. (the equation for ρ).

The interest in time - dependent systems has increased appreciably these last years. Several methods have been devised for the derivation of the Lewis invariant, which was originally obtained through an application of the assymptotic theory of Kruskal [5] in closed form : Leach [6] has obtained the same result using a time - dependent canonical transformation. Lutzky's [7] derivation was based on Noether's theorem. Ray and Reid [8] have resurrected the old Ermakov technique, and were able to obtain the existence of a Lewis - type invariant for the case of two coupled non linear equations of motion :

$$\ddot{x} + \omega^2(t)\, x = \frac{1}{x^2 \rho^2}\, g\left(\frac{\rho}{x}\right)$$

$$\ddot{\rho} + \omega^2 (t) = \frac{1}{\rho^2 \dot{x}} f\left(\frac{x}{\rho}\right)$$

namely $C = \frac{1}{2} (x\dot{\rho} - \rho\dot{x})^2 + \int \frac{x}{\rho} f(\eta)\, d\eta + \int \frac{\rho}{x} g(\eta)\, d\eta.$

In a series of papers, Ray, Reid and Lutzky [9 - 10] have extended further the class of non linear equations which possess an exact invariant. They have shown how the same results can be reached using Noether's theorem and demonstrated that there exists a general, non linear superposition law for the systems they studied.

A particularly simple analysis, which provides an insight into the results of Ray, Reid and Lutzky, has been given by Sarlet [17] who has relate the existence of the invariant to the integrability of a certain differential one - form. In a more recent paper, Sarlet and Ray [18] have provided a classification scheme for Ermakov type differential systems, establishing thus some unity into the multitude of examples of time - dependent systems with an exact invariant treated in the litterature.

In the Ermakov methodology, one derives the invariant starting from a set of given equations i.e. the auxiliary equation must be known in advance. However, when one starts from an explicitly time - dependent equation of motion, there is no simple way to guess even the existence of such an auxiliary equation, let alone its form. Because of this and the fact that the method of symmetry transformations, based for example on Noether's theorem, can be roundabout, Lewis and Leach [19] have presented a direct approach for the determination of the invariant of the system with a Hamiltonian of the form $H = \frac{1}{2} p^2 + V(x, t)$.

The extension of the above results to several spatial dimensions presents, of course, a great interest. Some results exist in this direction, although not as ample as in the case of one dimension due to the complexit of the problem. Günther and Leach [20] have derived a tensor invariant for an N - dimensional time - dependent isotropic harmonic oscillator. Ray and Reid [22] as well as Lutzky [9] have given a brief discussion concerning the extension of their method to several spatial dimensions. In a more recent work, Sarlet and Cantrijn [21] have presented a generalization of this

method which, in principle, deals with systems of n + 1 second order differential equations with n first integrals quadratic in the velocities. As in the case of the Ermakov systems, one of the equations plays the role of the auxiliary equation.

In the present work, we will present a study of two dimensional time - dependent Hamiltonian systems from the point of view of the existence of an exact invariant. The method used is a natural extension of our previous work on completely integrable (time - independent) Hamiltonian systems in two dimension [23]. However, the explicit time dependence of the potential will modify the calculations appreciably. We will use a direct approach for the construction of an invariant polynomial in the velocities of degree one or two. From this respect, our work constitutes an extension in two dimensions of the work of Lewis and Leach on one dimensional time - dependent Hamiltonians.

In the second and third sections of this paper, we present the construction of linear and quadratic invariants, respectively. In the fourth section, a comparison with previous results is presented, together with our conclusion.

II - CONSTANTS LINEAR IN THE VELOCITIES

We will consider a Hamiltonian of the form :

$$(1) \quad H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + V(x, y, t)$$

The equations of motion associated to this system are simply

$$(2) \quad \ddot{x} = - V_x , \quad \ddot{y} = - V_y ,$$

and we can notice that, as the potential V depends explicitly on the time t, H is not a constant of the motion. We will first concentrate on the search of an invariant linear in the velocities. It has the general form :

$$(3) \quad C = g^0 \dot{x} + g^1 \dot{y} + h.$$

Where g^0, g^1 and h are functions of x, y and t.

The condition $\frac{dC}{dt} = 0$ leads to the following polynomial identity in terms of $\dot{x}$ and $\dot{y}$:

$$(4) \quad g^0{}_x \dot{x}^2 + g^1{}_y \dot{y}^2 + (g^0{}_y + g^1{}_x) \dot{x}\dot{y} + (g^0{}_t + h_x) \dot{x} + (g^1{}_t + h_y) \dot{y} + h_t + g^0 \ddot{x} + g^1 \ddot{y} = 0.$$

This is equivalent to equating to zero the coefficient of each distinct monomial in $\dot{x}$ and $\dot{y}$ and leads to :

$$(5) \quad \begin{cases} g^0{}_x = 0 \\ g^0{}_y + g^1{}_x = 0 \\ g^1{}_y = 0 \end{cases}$$

$$(6) \quad \begin{cases} g^0{}_t + h_x = 0 \\ g^1{}_t + h_y = 0 \end{cases}$$

$$(7) \quad h_t + g^0 \ddot{x} + g^1 \ddot{y} = 0$$

The integration of the system (5) is straightforward and reads :

$$(8) \quad \begin{cases} g^0 = \alpha(t)\, y + \beta(t) \\ g^1 = -\alpha(t)\, x + \gamma(t) \end{cases}$$

The system (6) leads to a compatibility condition (9) which ensures the existence of the function h :

$$(9) \quad g^0{}_{t,y} = g^1{}_{t,x},$$

or in terms of α, β, γ :

$$2\alpha'(t) = 0.$$

Thus, since α is time - independent, we can easily integrate the equations (6) and obtain for h :

$$(10) \quad h = -x\beta'(t) - y\gamma'(t) + \varepsilon(t)$$

The last relation (7) reads, in terms of α, β, γ, ε :

$$(11)\quad x\,\beta''(t) + y\,\gamma''(t) - \varepsilon'(t) + (\alpha y + \beta)\,V_x - (\alpha x - \gamma)\,V_y = 0.$$

Equation (11) is the linear P.D.E. that the potential V must satisfy for the system to possess an invariant linear in the velocities.

We will distinguish two cases :

a) $\alpha = 0$. The equation (11) reduces to :

$$\beta\,V_x + \gamma\,V_y + \beta''x + \gamma''y - \varepsilon' = 0$$

or equivalently :

$$(12)\quad V_\xi + V_\eta + \frac{\beta''}{\beta}\xi + \frac{\gamma''}{\gamma}\eta - \varepsilon' = 0$$

with $\xi = \frac{x}{\beta}$, $\eta = \frac{y}{\gamma}$.

The integration of the homogeneous equation :

$V_\xi + V_\eta = 0$ is straightforward and reads :

$$(13)\quad V = F(\xi - \eta, t)$$

We need now a particular solution of equation (12). For this, we introduce the following variables :

$$u = \xi - \eta$$
$$v = \eta + \xi,$$

which lead to the following form of equation (12) :

$$(14)\quad 2V_v - Bv - \Gamma u - \varepsilon' = 0$$

with $\Gamma = -\frac{1}{2}\left(\frac{\beta''}{\beta} - \frac{\gamma''}{\gamma}\right)$, $B = -\frac{1}{2}\left(\frac{\beta''}{\beta} + \frac{\gamma''}{\gamma}\right)$

It is immediate to check that

$V = \frac{1}{2}\left(\Gamma\,u\,v + \frac{1}{2}\,B\,v^2 + \varepsilon'\,v\right)$ is a solution of (14), and thus the genera solution of equation (12) reads :

(15) $V = F(\xi - \eta, t) + \frac{1}{2}\left[\left(\Gamma + \frac{1}{2} B\right) \xi^2 - \left(\Gamma - \frac{1}{2} B\right) \eta^2 + B \xi \eta + \varepsilon' \xi + \varepsilon' \eta\right]$

Let us now examine the second case :

b) $\alpha \neq 0$. In terms of the variables :

$\xi = x - \frac{\gamma}{\alpha}$, $\eta = y + \frac{\beta}{\alpha}$, equation (11) becomes :

(16) $\eta V_\xi - \xi V_\eta + \frac{\beta''}{\alpha} \xi + \frac{\gamma''}{\alpha} \eta + \frac{\beta''}{\alpha^2} \gamma - \frac{\beta \gamma''}{\alpha^2} - \frac{\varepsilon'}{\alpha} = 0.$

Transforming into polar coordinates, $\xi = \rho \cos \varphi$, $\eta = \rho \sin \varphi$, it takes the simpler form :

$$V_\varphi = \frac{\beta''}{\alpha} \rho \cos \varphi + \frac{\gamma''}{\alpha} \rho \sin \varphi + \frac{\beta'' \gamma - \beta \gamma''}{\alpha^2} - \frac{\varepsilon'}{\alpha}$$

Its general solution is thus :

$$V = F(\rho) + \frac{\beta''}{\alpha} \rho \sin \varphi - \frac{\gamma''}{\alpha} \rho \cos \varphi + A \varphi,$$

or in terms or ξ an η,

(17) $V = \frac{\beta''}{\alpha} \eta - \frac{\gamma''}{\alpha} \xi + A \operatorname{arctg}\left(\frac{\eta}{\xi}\right) + F(\xi^2 + \eta^2, t)$

with $A = \frac{\beta'' \gamma - \beta \gamma''}{\alpha^2} - \frac{\varepsilon'}{\alpha}$.

Formula (15) and (17) exhaust all the possible forms of potential for which a constant linear in the velocities exists.

III - CONSTANTS QUADRATIC IN THE VELOCITIES :

The general form of such a constant is :

(18) $C = f^0 \dot{x}^2 + f^1 \dot{x} \dot{y} + f^2 \dot{y}^2 + g^0 \dot{x} + g^1 \dot{y} + h.$

Following the method of section II, we write $\frac{dC}{dt}$ as a polynomial of degree 3 in $\dot{x}$ and $\dot{y}$:

$$(19)\quad \frac{dC}{dt} = f^0{}_x\,\dot{x}^3 + (f^0{}_y + f^1{}_x)\,\dot{x}^2\,\dot{y} + (f^1{}_y + f^2{}_x)\,\dot{x}\,\dot{y}^2 + f^2{}_y\,\dot{y}^3 +$$
$$(g^0{}_x + f^0{}_t)\,\dot{x}^2 + (g^0{}_y + f^1{}_t + g^1{}_x)\,\dot{x}\,\dot{y} + (f^2{}_t + g^1{}_y)\,\dot{y}^2 -$$
$$(g^0{}_t + h_x + 2\,f^0\,\ddot{x} + f^1\,\ddot{y})\,\dot{x} + (g^1{}_t + h_y + f^1\,\ddot{x} + 2\,f^2\,\ddot{y})\,\dot{y} +$$
$$(h_t + g^0\,\ddot{x} + g^1\,\ddot{y}).$$

We are thus led to systems of partial differential equations for the f_i, g_i and h.

The first set of equations for the f_i's can be easily integrated :

$$(20)\quad \begin{cases} f_0 = \alpha y^2 + \beta y + \gamma \\ f_1 = -2\,\alpha\,xy - \beta\,x - \delta\,y - \varepsilon \\ f_2 = \alpha\,x^2 + \delta\,x + \zeta \end{cases}$$

The functions f_i have the same quadratic dependence in x and y as in the case of time independent Hamiltonian [22]. The main difference, here, stems from the fact that the coefficients depend explicitly on time. The constant C (18) can also be written in terms of the angular momentum $L = x\dot{y} - y\dot{x}$:

$$(21)\quad C = \alpha\,L^2 - \beta\,\dot{x}\,L + \delta\,\dot{y}\,L + \gamma\,\dot{x}^2 - \varepsilon\,\dot{x}\,\dot{y} + \zeta\,\dot{y}^2 + g^0\,\dot{x} + g^1\,\dot{y} + h.$$

The remaining equations have the form :

$$(22)\quad \begin{cases} g^0{}_x + f^0{}_t = 0 \\ g^0{}_y + f^1{}_t + g^1{}_x = 0 \\ g^1{}_y + f^2{}_t = 0 \end{cases}$$

$$(23)\quad \begin{cases} g^0{}_t + 2\,f^0\,\ddot{x} + f^1\,\ddot{y} + h_x = 0 \\ g^1{}_t + f^1\,\ddot{x} + 2\,f^2\,\ddot{y} + h_y = 0 \end{cases}$$

$$(24)\quad h_t - g^0\,\ddot{x} - g^1\,\ddot{y} = 0$$

From the knowledge of the functions f_i, system (22) allows the calculation of the functions g_i, providing the following compatibility condition is satisfied :

$$\left(f^0_{yy} - f^1_{xy} + f^2_{xx} \right)_t = 0$$

Due to the special form (20) of the functions f_i, this condition reduces to $\alpha'(t) = 0$. Once the g_i's are known, a second compatibility condition, which allows the calculation of h, results from the system (23) :

$$(25) \quad \frac{\partial}{\partial y} \left(g^0_t + 2 f^0 \ddot{x} + f^1 \ddot{y} \right) = \frac{\partial}{\partial x} \left(g^1_t + f^1 \ddot{x} + 2 f^2 \ddot{y} \right)$$

That is :

$$(26) \quad f_1 (V_{yy} - V_{xx}) + 2 (f_0 - f_2) V_{xy} + (2 f^0_y - f^1_x) V_x - (2 f^2_x - f^1_y) V_y = g^0_{ty} - g^1_{tx}$$

This last relation is quite similar to the one obtained in the search of a time independent potential V, that admits a constant of motion quadratic in the velocities [22]. The difference is only in the existence of a non homogeneous part in this linear P.D.E. This remark will lead us to the same classification as in the autononous case. Before proceeding further, let us point out that, once V is determined satisfying (26), there remains a last relation (24) to check for the system to possess a quadratic invariant. This was not the case for time independent potentials and, as we will see further, this relation strongly reduces the admissible forms of potentials.

We will distinguish three distinct cases, according to the value of the highest power of the angular momentum L that appears in the constant (21). In each case, we will reduce the form of the invariant by translations and rotations of coordinates. One can note that a rotation does not change L, while a translation keeps $\dot{x}$ and $\dot{y}$ invariant.

Case a $\alpha = \beta = \delta = 0$. This the separable case.

There is no dependence in L in the constant C.

By an adequate rotation, the coefficient of $\dot{x}\,\dot{y}$ can be set to zero unless $(\gamma - \zeta)^2 + \varepsilon^2 = 0$.

Case b $\alpha = 0$, β (or δ) non vanishing.

Translations of x and y allow us to eliminate $\gamma - \zeta$ and ε and an adequate rotation of coordinates allows the choice $\delta = 0$, unless $\beta^2 + \delta^2 = 0$ as can be easily seen from the form (21) of the constant C.

Case c $\alpha \neq 0$.

Translation of x and y allow the elimination of all the linear in L terms in (21). ($\beta = \delta = 0$). Then, by a rotation ζ can be set equal to zero unless $\varepsilon^2 + (\gamma - \zeta)^2 = 0$.

We will now proceed with the integration of equations (22), (23) and (24) in each of the distinct reduced cases a, b, and c.

Case a : $f_0 = \gamma$, $f_1 = 0$, $f_2 = \zeta$.

The integration of equations (22) leads to :

$$(27) \qquad \begin{aligned} g_0 &= -\gamma' x + \Theta y + \lambda \\ g_1 &= -\zeta' y - \Theta x + k \end{aligned}$$

In the following, we will choose $\lambda = k = 0$; it corresponds to an adequate translation of coordinates. However, with $\Theta \neq 0$, there exists no solution to the system other than the trivial harmonic oscillator : $V = \frac{1}{2}\,\Phi(t)\,x^2 + \frac{1}{2}\,\Psi(t)\,y^2$. Apparently, the condition $\Theta \neq 0$ imposes severe constraints on the potential. For this reason, we will look for solutions with $\Theta = 0$.

The condition (25) writes :

$2(\gamma - \zeta)\,V_{xy} = 0$, or

(28) $V = F(x, t) + G(y, t)$

Integration of equation (23) for h is straightforward and leads to :

$$(29)\quad h = 2\gamma F + 2\zeta G + \frac{1}{2}\gamma'' x^2 + \frac{1}{2}\zeta'' y^2$$

And finaly, equation (24) takes the following form :

$$2\gamma' F + 2\zeta' G + 2\gamma F_t + 2\zeta G_t + \frac{1}{2}\gamma''' x^2 + \frac{1}{2}\zeta''' y^2 = -\gamma' x F_x - \zeta' y G_y .$$

This equation separates (up to a function of time, to be included in F or G) into the system :

$$(30)\quad \begin{aligned} 2\gamma' F + 2\gamma F_t + \gamma' x F_x &= -\frac{1}{2}\gamma''' x^2 \\ 2\zeta' G + 2\zeta G_t + \zeta' y G_y &= -\frac{1}{2}\zeta''' y^2 \end{aligned}$$

In order to solve the equation for F, we look for a particular solution of the form : $F(x) = x^2 \Psi(t)$. It leads to :

$\Psi' + 2\frac{\gamma'}{\gamma}\Psi = -\frac{1}{4}\frac{\gamma'''}{\gamma}$, whose solution writes :

$$\Psi(t) = -\frac{1}{4}\left(\frac{\gamma''}{\gamma} - \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)^2\right)$$

The general solution of (30) is now easy to obtain ; it writes :

$$(31)\quad F(x, t) = \frac{1}{\gamma}\chi\left(\frac{x}{\sqrt{\gamma}}\right) + x^2\psi(t) \quad .$$

The expression of G is similar in terms of ζ and y, instead of γ and x

We will now investigate the particular case $\varepsilon^2 + (\gamma - \zeta)^2 = 0$, where the rotation is not possible.

One obtains for f_i and g_i :

$f_0 = \gamma$, $f_1 = i(\gamma - \zeta)$, $f_2 = \zeta$, $g^0 = -\gamma' z$, $g^1 = i\zeta' z$

(with the integration constants taken equal to zero as in the case a)abo

with $z = x + iy$

The partial differential equation for V (26), has the form :

$$\varepsilon V_{yy} - V_{xx} - 2i\varepsilon V_{xy} = -i\sigma''$$

with $\sigma = \zeta + \gamma$, $\varepsilon = i(\gamma - \zeta)$.

or in terms of z and $\bar{z} = x - iy$,

$$-4i\varepsilon V_{z\bar{z}} = \sigma''$$

The integration for V is thus trivial :

$$(32) \quad V = i\frac{\sigma''}{8\varepsilon}\bar{z}^2 + \bar{z} F_z(z, t) + G(z, t)$$

The system of P.D.E. for h, in terms of z and $\bar{z}$ reads :

$$(33) \quad \begin{aligned} h_z &= \tfrac{1}{2}\left[-g^0{}_t + ig^1{}_t + V_z(2f_0 + 2f_2) + V_{\bar{z}}(2f_0 - 2f_2 - 2if_1)\right] \\ h_{\bar{z}} &= \tfrac{1}{2}\left[-g^0{}_t - ig^1{}_t + V_z(2f_0 - 2f_2 + 2if_1) + V_{\bar{z}}(2f_0 + 2f_2)\right] \end{aligned}$$

In this precise case, due to the values of g and f, it reduces to :

$$h_z = -i\varepsilon''\frac{z}{2} + \sigma V_z - 2i\varepsilon V_{\bar{z}}$$

$$h_{\bar{z}} = \sigma''\frac{z}{2} - \sigma V_{\bar{z}} \quad ,$$

leading thus to :

$$(34) \quad h = \sigma V + \frac{z\bar{z}}{2}\sigma'' - i\varepsilon''\frac{z^2}{4} - 2i\varepsilon F$$

The last relation is now the following :

$$h_t = -\sigma' z(\bar{z} F_z + G_z) - \mu' z\left(\frac{\sigma''}{4\mu}\bar{z} + F\right)$$

Separating the different terms according to their dependance on $\bar{z}$ in this last relation, we obtain :

$$(35) \quad \left(\frac{\sigma\sigma''}{\varepsilon}\right)' = 0 \qquad \text{(terms in } \bar{z}^2\text{). Hence} \qquad \frac{\sigma\varepsilon''}{\varepsilon} = 4\nu \text{ (cnst.)}$$

$$(36) \quad \sigma' F_z + \sigma F_{zt} + \sigma' z F_{zz} = -\frac{z}{4}\left(2\sigma''' + \frac{\varepsilon'\sigma''}{\varepsilon}\right) \text{ (terms in } \bar{z}^1\text{)}$$

$$(37) \quad \sigma' G + \sigma G_t + \sigma' z G_z = -\varepsilon''' \frac{z^2}{4} - 2\varepsilon' F - 2\varepsilon F_t - \varepsilon' z F_z \text{ (terms in } \bar{z}^0\text{)}$$

One recognizes readily in (36), (37), the same left hand side as in Eq. (30). Integrating, we obtain for F :

$$(38) \qquad F = \Phi\left(\frac{z}{\sigma}\right) - \frac{1}{2} m z^2$$

where $m = \frac{1}{2}\frac{\sigma''}{\sigma} - \frac{1}{4}\left(\frac{\sigma'}{\sigma}\right)^2 + \frac{\nu\varepsilon}{\sigma^2}$

Once F is known, it is easy to check that the right hand side of (37) is of the form : $n z^2 - 2\varepsilon'\Phi + \frac{1}{\sigma}(2\varepsilon\sigma' - \varepsilon')\, z\Phi'$, where $n = -\frac{\mu'''}{4} - 2\mu' m - \mu m'$.

The solution of (37) is thus the following :

$$(39) \qquad G = \frac{1}{\sigma}\Psi\left(\frac{z}{\sigma}\right) + pz^2 - \frac{2\varepsilon}{\sigma}\Phi(qz)$$

where $p = -\sigma^{-3}\int \eta\sigma^2 dt$, $q = \frac{1}{\sigma}\ln\frac{\sqrt{\varepsilon}}{\sigma}$.

(The case of power-like Φ should in principle be treated apart).

Relations (32), (38) and (39) determine the precise form of a potential V for which a constant of that kind exists.

Let us proceed now with the search of a constant linear in L.

Case b :

In that case, we obtain for f_i and g_i :

$$(40) \qquad \begin{cases} f^0 = \beta y + \gamma \\ f^1 = -\beta x \\ f^2 = \gamma \end{cases}$$

$$(41) \qquad \begin{cases} g^0 = -(\beta' y + \gamma')\, x + \theta y + \lambda \\ g^1 = -\gamma' y + \beta' x^2 - \theta x + k \end{cases}$$

The analysis of the corresponding case for time independent Hamiltonians has shown that the adequate variables were :

$$u = \rho + \eta \qquad v = \rho - \eta \qquad \text{where } \rho = \sqrt{x^2 + y^2},\ \eta = y.$$

Moreover, a detailed analysis has shown that there is no solution with non vanishing λ and k and we will thus put them to zero in order to alleviat

the presentation. In terms of u and v, equations (23) take the following form :

$$
\begin{aligned}
h_u &= (-\beta v + 2\gamma)\, V_u + \frac{1}{4}\,(-\beta'' v + \gamma'')\,(u+v) + \frac{1}{4}\,\frac{\theta' v\,(u+v)}{\sqrt{u\,v}} \\
h_v &= (\beta u + 2\gamma)\, V_u + \frac{1}{4}\,(\beta'' u + \gamma'')\,(u+v) - \frac{1}{4}\,\frac{\theta' u\,(u+v)}{\sqrt{u\,v}}
\end{aligned}
\tag{42}
$$

The compatibility equation resulting from (42) leads to a P.D.E. for V in terms of the independent variables u and v. Its solution is straightforward :

$$
V = \frac{F(u) + G(v)}{u+v} - \frac{3\,\beta''}{8\,\beta}\,u\,v + \frac{2\,\theta'}{3\,\beta}\,\sqrt{u\,v}\,, \tag{43}
$$

as well as the integration of (42) :

$$
\begin{aligned}
h = {} & \beta\,\frac{u\,G(v) - v\,F(u)}{u+v} + 2\gamma\,\frac{F(u) + G(v)}{u+v} + \frac{1}{8}\,\beta''\,uv\,(v-u) \\
& + \frac{1}{8}\,\gamma''\,(u+v)^2 - \frac{3}{4}\,\frac{\gamma\,\beta''uv}{\beta'} + \frac{4\,\gamma\,\theta'}{3\,\beta}\,\sqrt{uv} + \frac{1}{6}\,\theta'\,(u-v)\,\sqrt{uv}
\end{aligned}
\tag{44}
$$

There remains now equation (24) to be checked. This relation will impose constraints on the form of the potential V and on the time dependent functions β, γ, θ. Three different kinds of terms, functions of u alone, v alone as well as terms where u and v are mixed together, are involved. These three families of functions will give three distinct relations. After some lengthy manipulations, it results, from the mixed term relation, that θ and β'' must vanish. In this case, this last relation reduces to :

$$
\begin{aligned}
G + \frac{\beta}{\beta'}\,G_t + v\,G_v &= 0 \\
F + \frac{\beta}{\beta'}\,F_t + u\,F_u &= 0
\end{aligned}
\tag{45}
$$

On the other hand, the terms that depend only on u in the relation (24) lead to the following constraint :

$$
F + \frac{2\,\gamma}{\gamma'}\,F_t + u\,F_u = 0,
$$

and similarly for G :

$$
G + \frac{2\,\gamma}{\gamma'}\,G_t + v\,G_v = 0
$$

All these equations are compatible only if $\gamma = \beta^2$.

We recognize in (45), Equations (30) up to the right hand side and we thus obtain the solution of (45) :

$$F(u, t) = \frac{1}{\beta} \Phi\left(\frac{u}{\beta}\right)$$
$$G(v, t) = \frac{1}{\beta} \Psi\left(\frac{v}{\beta}\right)$$

In conclusion, the potential :

$$(46) \quad V = \frac{\Phi\left(\frac{y + \sqrt{x^2 + y^2}}{\beta}\right) + \Psi\left(\frac{\sqrt{x^2 + y^2} - y}{\beta}\right)}{\beta \sqrt{x^2 + y^2}}, \quad \beta = c_1 + c_2 t$$

is the general form of potential for which a constant of motion, linear in the angular momentum exists. The particular case where $\delta = i\beta$ must be treated apart.

One obtains, in this case, for f_i and g_i :

$$\begin{cases} f_0 = \beta y + \gamma \\ f_1 = -\beta x - i\beta y = -\beta z \\ f_2 = i\beta x + \gamma \end{cases}$$

$$\begin{cases} g_0 = -\beta' y\, \bar{z} - \gamma' x \\ g_1 = \beta' x\, \bar{z} - \gamma' y \end{cases}$$

with $z = x + iy$, $\bar{z} = x - iy$.

The partial differential equation for V, (26), in terms of z and $\bar{z}$ reduces to :

$$(47) \quad 2z\, V_{zz} + 3V_z = -\frac{3}{2} \frac{\beta''}{\beta} \bar{z}$$

Integrating (47), one finds :

$$(48) \quad V = z^{-\frac{1}{2}} F(\bar{z}, t) + G_{\bar{z}}(\bar{z}, t) - \frac{1}{2} \frac{\beta''}{\beta} z\bar{z}$$

Its is now easy to obtain from (33) the equations for h :

$$\begin{cases} h_z = \frac{1}{2}(\gamma'' + i\beta''\bar{z})\bar{z} + (i\beta\bar{z} + 2\gamma) V_z \\ h_{\bar{z}} = \frac{1}{2}(\gamma'' - i\beta''\bar{z}) z - 2i\beta z V_z + (i\beta\bar{z} + 2\gamma) V_{\bar{z}} , \end{cases}$$

from which we deduce the value of h :

$$(49) \quad h = (i\beta\bar{z} + 2\gamma) V + \frac{1}{2}(\gamma'' + i\beta''\bar{z}) z\bar{z} - i\beta G(\bar{z}, t)$$

As preceedingly, there remains a last relation (24) to be verified. It involves three distinct and independent families of terms ; namely functions of $\bar{z}$ that multiply either $z^{-1/2}$, z or 1. We thus obtain the following three equations :

$$2\gamma' \frac{\beta''}{\beta} + \gamma \left(\frac{\beta''}{\beta}\right)' - \frac{1}{2}\gamma''' = 0$$

$$\frac{3}{2}(i\beta'\bar{z} + \gamma') F + (i\beta\bar{z} + 2\gamma) F_t + (\gamma' + i\beta'\bar{z})\bar{z} F_{\bar{z}} = 0$$

$$G_{\bar{z}t}(i\beta\bar{z} + 2\gamma) + G_{\bar{z}}(i\beta'\bar{z} + 2\gamma') + (i\beta'\bar{z} + \gamma')\bar{z} G_{\bar{z}\bar{z}} = i\beta' G + i\beta G_t$$

By analogy to the general case b), we will look for solutions with $\gamma = 0$. Indeed in that case, equations reduce to :

$$(50) \qquad \frac{3}{2}\beta' F + \beta F_t + \beta'\bar{z} F_{\bar{z}} = 0$$

$$(51) \qquad \bar{z}(\beta G_{t\bar{z}} + \beta' G_{\bar{z}} + \beta'\bar{z} G_{\bar{z}\bar{z}}) = \beta' G + \beta G_t$$

and we recognize the form of the equation (45) for Eq. (50).

The general solution for F reads thus :

$$(52) \qquad F = \frac{1}{\beta^{3/2}} \chi\left(\frac{\bar{z}}{\beta}\right)$$

Concerning the equation for G, the following change of the dependant variable :

$$H = \bar{z} G_{\bar{z}} - G$$

leads to the expression :

$$\beta' H + \beta H_t + \beta' \bar{z} H_{\bar{z}} = 0$$

the solution of which reads :

$$H = \frac{1}{\beta} \Omega\left(\frac{\bar{z}}{\beta}\right)$$

or equivalently,

$$H = \frac{\bar{z}^2}{\beta} \frac{d}{d\bar{z}} \left(\frac{1}{\bar{z}} \Phi\left(\frac{\bar{z}}{\beta}\right)\right) \tag{53}$$

and finally :

$$G = \frac{1}{\beta} \Phi\left(\frac{\bar{z}}{\beta}\right) + \bar{z}\, \psi(t)$$

In conclusion, provided V is given in terms of z and $\bar{z}$ by formula (48), (52) and (53), a constant of the motion, linear in L exists with the condition $\beta = i\delta$.

Case c :

The last case to examine is the case where the constant C has a quadratic dependence in L, namely, after the necessary reductions, the constant (21)assumes the form :

$$C = \alpha L^2 + \gamma \dot{x}^2 + \zeta\, \dot{y}^2 + g_0\, \dot{x} + g_1\, \dot{y} + h.$$

As we have seen before, (9), α must be time independent and will be taken equal to 1.

We first examine the case $\gamma = \zeta$.

The corresponding values of the f_i's and g_i's write :

$$\begin{cases} f_0 = y^2 + \gamma \\ f_1 = -2xy \\ f_2 = x^2 + \gamma \end{cases}$$

$$\begin{cases} g_0 = -\gamma' x + \lambda y \\ g_1 = -\gamma' y - \lambda x \end{cases}$$

The P.D.E. for V is easily solved using polar coordinates and leads to the following form for V :

$$(54) \qquad V = \frac{F\ (\Theta,\ t)}{\rho^2} + G\ (\rho,\ t) - \Theta\ \frac{\lambda'\ (t)}{2}$$

$$\Theta = \operatorname{arctg} \frac{y}{x}\ ,\quad \rho^2 = x^2 + y^2\ ,$$

while h writes :

$$(55) \qquad h = 2\ \gamma\ \frac{F}{\rho^2} + 2\ F + 2\ \gamma\ G - \gamma\ \lambda'\ \Theta + \frac{\gamma''\ \rho^2}{2}$$

The last compatibility relation, in terms of ρ, Θ and t reads :

$$h_t = - \gamma'\rho\ V_\rho - \lambda\ V_\Theta .$$

Again, three different kinds of terms appear which must vanish separately : this leads to the following relations :

$$2\ F_t - \Theta\gamma'\lambda' - \gamma\Theta\lambda'' = \frac{\lambda\lambda'}{2} .$$

$$\frac{\gamma'''}{2}\ \rho^2 + 2\ \gamma'\ G + 2\ \gamma\ G_t = - \gamma'\rho\ G_\rho$$

$$2\ \gamma\ F_t = - \lambda\ F_\Theta$$

The equation for G has been encountered previously. Its solutions is :

$$(56) \qquad G\ (\rho,\ t) = \frac{1}{\gamma}\ \varphi\ \left(\ \frac{\rho^2}{\gamma}\ \right) - \frac{\rho^2}{4}\ \left(\ \frac{\gamma''}{\gamma} - \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)^2\right)$$

Now, the two distinct equations for F put constraints on γ and λ.

A solution (57) $F = \frac{\lambda^2}{8} - \frac{\Theta k}{2}$ exists whenever $k = \gamma\ \lambda'$ is time independent.

The general solution V is thus the following :

$$(58) \qquad V = \frac{1}{\gamma}\ \varphi\ \left(\ \frac{x^2 + y^2}{\gamma}\ \right) - \frac{x^2 + y^2}{4}\ \left(\ \frac{\gamma''}{\gamma} - \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)^2\right)$$

$$+ \frac{1}{x^2+y^2}\left[\ \frac{\lambda^2}{8} - \frac{k}{2}\ \operatorname{Arctg}\ \frac{y}{x}\ \right] - \frac{k}{2\ \gamma}\ \operatorname{Arctg}\ \frac{y}{x}$$

with $\lambda = k \int_0^t \frac{d\tau}{\gamma}$

When $\lambda = 0$, the solution is slightly different, because F can be any time independent function of θ.

(59) if $\gamma = 0$, one has $V = \frac{F(\theta)}{\rho^2} + G(\rho, t)$

(60) if $\gamma \neq 0$, one has $V = \frac{F(\theta)}{\rho^2} + \frac{1}{\gamma}\varphi\left(\frac{\rho^2}{\gamma}\right) - \frac{\rho^2}{4}\left(\frac{\delta''}{\delta} - \frac{1}{2}\left(\frac{\delta'}{\delta}\right)^2\right)$

We have also examined in detail the case $\gamma \neq$, but it failed to yield any solution. Thus we will not exhibit here any calculations. This completes the study of the time - dependent Hamiltonians with an exact invariant quadratic in the velocities.

IV - COMPARISON WITH OTHER RESULTS

As was stated in the introduction, there exist numerous studies on one - dimensional time - dependent systems but significantly fewer on two (or more) dimensional ones. Moreover, as the Ermakov approach is most often employed, the equations of motion are usually not Hamiltonian. Still some comparisons with existing results can be made.

To start with, our separable case should encompass the results for one - dimensional systems of Lewis and Leach [19]. They dealt with time dependent Hamiltonian Systems in one dimension $H = \frac{1}{2}p^2 + V(x, t)$, and give in particular conditions for which a potential $V(x, t)$ possesses a constant quadratic in the velocity p. These conditions determine the precise form of V in terms of arbitrary functions of time, namely :

(61) $V(x, t) = -F(t)\,x + \frac{1}{2}\Omega^2(t)\,x^2 + \frac{1}{\rho^2}U\left(\frac{x - \upsilon}{\rho}\right)$

U is an arbitrary function of its argument and F, Ω^2, ρ, υ are arbitrary functions submitted to the following constraints :

$$\ddot{\rho} + \Omega^2(t)\rho - \frac{k}{\rho^3} = 0$$

$$\ddot{\upsilon} + \Omega^2(t)\upsilon = F(t)$$

In section II, we examined separable potentials : $V = F(x, t) + G(y, t)$. Thus, the two directions x and y decouple and we have in fact two one - dimensional Hamiltonians. The following form was found for $F(x, t)$ (and similarly for $G(y, t)$) :

$$F(x, t) = \frac{1}{\gamma} \varphi \left(\frac{x}{\sqrt{\gamma}} \right) - \frac{x^2}{4} \left(\frac{\gamma''}{\gamma} - \frac{1}{2} \left(\frac{\gamma'}{\gamma} \right)^2 \right)$$

This form is identical to (61) up to a translation of x, which would correspond to a choice of $\lambda \neq 0$ in g_0.

We turn now to genuine two dimensional systems. In a recent publication, Lutzky [9] proved that the quantity :

$$C = \frac{1}{2} (x\dot{y} - \dot{x}y)^2 + \int_0^{\frac{x}{y}} \Psi(\lambda) \frac{d\lambda}{\lambda}$$

is conserved by the motion described by the equations :

$$(62) \qquad \begin{cases} \ddot{x} + \omega^2(t)\, x = f_1(x, y) \\ \ddot{y} + \omega^2(t)\, y = f_2(x, y) \end{cases}$$

provided that f_1 and f_2 satisfy the relation :

$$(63) \qquad x f_2 - y f_1 = \frac{1}{xy} \Psi \left(\frac{x}{y} \right).$$

This constant C is quadratic in L and corresponds to the case c) we introduced in III. Moreover we have $\gamma = \zeta = 0$. We have found that the only form of potential V, in that case was given by (59). In order to compare our results to Lutzky's, we must examine under which conditions his equations derive from a Hamiltonian. This happens if there exists a function W such that $f_1 = W_x$, $f_2 = W_y$. Relation (62) is thus equivalent to

$$x W_y - y W_x = \frac{\Psi(x/y)}{xy},$$

which in polar coordinates reads :

$$\frac{\partial W}{\partial \Theta} = \frac{F'(\Theta)}{\rho^2}, \quad F \text{ determined in terms of } \Psi.$$

that is $W = \frac{F(\Theta)}{\rho^2} + G(\rho, t)$.

This result coïncides with (59).

So we conclude that Lutzky's result is identical to ours whenever the equations of motion (62) are Hamiltonian.

V - CONCLUSION

In this paper, we have presented an investigation of time - dependent Hamiltonians in two space dimensions from the point of view of the existence of an exact invariant. The method we have used was the direct computation of the invariant, which was employed in our previous work on 2-D time independent Hamiltonians, as well as in Lewis - Leach's work on 1-D systems. We were able to identify the forms of the time - dependent potential for which an invariant linear or quadratic in the velocities exists. Our results extend thus the results obtained previously by various groups to the case of Hamiltonian systems. In particular, they contain as a special case the Hamiltonians of Lewis and Leach and have a non zero overlap with Lutzky's results.

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CHAPITRE 8

QUANTIFICATION DE SYSTÈMES CLASSIQUES HAMILTONIENS INDÉPENDANTS DU TEMPS

Dans les sections précédentes, nous avons rencontré un grand nombre de systèmes dynamiques hamiltoniens intégrables, à deux degrés de liberté. En particulier, nous avons déterminé la classe de tous les systèmes hamiltoniens possédant deux constantes de mouvement indépendantes, quadratiques dans les vitesses. D'autres cas de potentiels polynomiaux intégrables ont été exhibés (avec une constante de degré quatre dans les vitesses). Il est aussi bien connu que des chaînes de N particules interagissant entre elles sur une droite peuvent être intégrables (Calogero, Toda) pour certains types d'interaction.

A la vue de tous ces systèmes intégrables au sens de Liouville (c'est-à-dire possédant autant d'intégrales que de degrés de liberté, indépendantes du temps, analytiques, monovaluées et en involution, i.e. telles que le crochet de Poisson de deux quelconques d'entre elles s'annule), une question se pose naturellement : quel est le comportement des systèmes quantiques qui leur sont associés ? Le cas bien connu de l'oscillateur harmonique laisserait présumer que si un système classique à N degrés de liberté est intégrable, il existe N opérateurs qui commutent avec l'opérateur associé à l'hamiltonien $\hat{H}$ et entre eux (ce qui signifie exactement que ce système est quantiquement intégrable). La vérification directe de cette conjecture n'est en général pas triviale pour plusieurs raisons comme nous allons le voir.

Plaçons-nous dans le cas d'un système classiquement intégrable à N dimensions. L'idée naturelle consiste à associer aux N constantes du mouvement (I_m) des opérateurs $\hat{I}_m$, mais alors qu'il n'y a qu'un seul moyen d'associer à un hamiltonien classique de la forme :

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + V(x_1,\ldots,x_N)$$

l'opérateur quantique :

$$\hat{H} = \frac{1}{2} \sum_{i=1}^{N} \hat{p}_i^2 + V(\hat{x}_1,\ldots,\hat{x}_N) \qquad \left(p_i = -i\hbar \frac{\partial}{\partial x_i}\right),$$

ce n'est généralement pas le cas pour les autres intégrales du mouvement $I_n(x,p)$ où peuvent apparaître des termes de la forme $f(p_i)$, $g(x_i)$ qui ne définissent pas un opérateur de manière unique en fonction de $\hat{p}_i$, $\hat{x}_i$.

Un moyen systématique de quantification est donnée par la règle de Weyl [1] ; cette règle permet de faire correspondre à la quantité classique $B(x,p)$ l'opérateur $\hat{B}$ dont les éléments de matrice sont donnés par :

$$\langle x/\hat{B}/x' \rangle = \int B(\tfrac{1}{2}(x+x'),p) e^{ip(x-x')/\hbar} \frac{dp}{(2\pi\hbar)N}$$

L'opération inverse est la transformée de Wigner. Elle fait correspondre à l'opérateur $\hat{A}$, la quantité définie par :

$$A_w(x,p,\hbar) = \int \langle x - \tfrac{1}{2}x'' /\hat{A}/ x + \tfrac{1}{2}x'' \rangle e^{ipx''/\hbar} dx''$$

Ces deux opérations inverses établissent une correspondance entre objets classiques et quantiques [1]. En particulier, au commutateur de deux opérateurs $\hat{A}$ et $\hat{B}$ est associée la quantité :

$$(\hat{A}\hat{B})_w - (\hat{B}\hat{A})_w \qquad (1)$$

La transformée de Wigner du produit de deux opérateurs $\hat{A}$, $\hat{B}$ se trouve être égale à :

$$(\hat{A}\hat{B})_w = A_w \exp\left[(\tfrac{1}{2} i\hbar \overleftrightarrow{\Lambda}\right] B_w$$

où :

$$\overleftrightarrow{\Lambda} = \sum_{i=1}^{N} \left(\frac{\overleftarrow{\partial}}{\partial x_i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial x_i}\right)$$

La quantité *(1)* divisée par $i\hbar$ définit le crochet de Moyal, lequel, à la limite classique, redonne le crochet de Poisson.

En conséquence, la vérification de la commutativité de deux opérateurs $\hat{A}$ et $\hat{B}$, définis à partir des quantités classiques A et B au moyen de la règle de Weyl, peut se faire directement sur A et B par le calcul de quantités du type $A\,\overleftrightarrow{\Lambda}^n B$.

D'autre part, on vérifie aisément que deux quantités classiques I et J, de crochet de Poisson nul, peuvent donner lieu, selon le type de quantification choisi, à des opérateurs qui commutent entre eux ou pas.

Un exemple simple en est donné par un cas particulier de l'hamiltonien de Hénon-Heiles :

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x^2y + 2y^3$$

pour lequel la seconde constante vaut :

$$C = x\dot{x}\dot{y} - y\dot{x}^2 + x^2y^2 + x^4/4$$

Si on quantifie C à l'aide de la règle de Weyl, on vérifie que $[\hat{H},\hat{C}] = 0$ alors que si on associe à C l'opérateur $\hat{\hat{C}} = xp_xp_y - yp_x^2 + x^2y^2 + x^4/4$, ce n'est plus le cas. Le choix d'une "bonne" quantification va donc être crucial pour notre problème.

Dans certains cas, il peut se faire aussi que les quantités classiques qui vont donner naissance à des opérateurs qui commutent ne soient pas exactement les constantes du mouvement classique mais une modification de celles-ci par des termes qui contiennent $\hbar$ explicitement.

Pour certains systèmes dynamiques à N degré de liberté (Toda, Calogéro)[2,3], on peut construire une paire de Lax, c'est-à-dire trouver deux matrices (L,B) de dim N×N telles que l'on puisse écrire les équations du mouvement sous la forme :

$$\frac{dL}{dt} = [B,L]$$

Les valeurs propres de la matrice L ou les fonctions symétriques de ces valeurs propres sont indépendantes du temps, et sont donc des intégrales du mouvement. Si on peut en trouver N en involution, le système considéré est intégrable. La connaissance de la matrice L permet dans certains cas de décrire certaines propriétés quantiques des opérateurs (I_k) associés aux intégrales du mouvement classique et se révèle donc un outil précieux dans l'étude du système tant classique que quantique.

Dans la suite de cette section, nous nous proposons d'illustrer par des exemples les propos précédents. Tout d'abord, nous montrerons que les potentiels $V(q_1,\ldots,q_N)$ qui possèdent N constantes de degré deux dans les vitesses,en involution sont aussi quantiquement intégrables, puis nous examinerons le cas de potentiels polynomiaux avec des constantes de degrés deux et trois dans les vitesses ; ceci, avant d'envisager les chaînes de Toda ou de Calogéro à N degrés de liberté.

1/ Considérons un système dynamique à deux degrés de liberté régi par l'hamiltonien suivant :

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x,y)$$

et tel qu'une seconde intégrale de la forme :

$$C = g_0p_x^2 + g_1p_xp_y + g_2p_y^2 + M(x,y)$$

existe. On a donc :

$$\{H,C\}_p = 0 = H \wedge C$$

où $\{H,C\}_p$ désigne le crochet de Poisson.

La transformée de Weyl nous permet de définir des opérateurs $\hat{H}$ et $\hat{C}$.

Il reste à vérifier que $[\hat{H},\hat{C}] = 0$.

Pour cela, il suffit de calculer :

$$(\hat{H}\hat{C})_w - (\hat{C}\hat{H})_w = 2i\, H\left(\sin \frac{\hbar}{2} \overleftrightarrow{\Lambda}\right) C$$

On développe : $\sin \frac{\hbar}{2} \overleftrightarrow{\Lambda} = \frac{\hbar}{2} \overleftrightarrow{\Lambda} - \left(\frac{\hbar}{2}\right)^3 \frac{\overleftrightarrow{\Lambda}^3}{3!} + \ldots$

Le calcul de Λ^3, étant donné la forme de l'hamiltonien H, se réduit à :

$$H\, \overleftrightarrow{\Lambda}^3 C = H\left(\overleftarrow{\partial}_{x^3} \overrightarrow{\partial}_{p_x^3} + \overleftarrow{\partial}_{y^3} \overrightarrow{\partial}_{p_y^3} + 3\overleftarrow{\partial}_{x^2y} \overrightarrow{\partial}_{p_x^2 p_y} + 3\overleftarrow{\partial}_{xy^2} \overrightarrow{\partial}_{p_y^2 p_x}\right) C$$

Comme C ne contient pas de terme en (p_x, p_y) de degré supérieur à 2, $H\, \Lambda^3\, C$ est identiquement nul ; et de manière analogue, tous les termes $H\, \Lambda^k\, C$, $k \geqslant 3$, seront nuls ; par conséquent :

$$(\hat{H}\hat{C})_w - (\hat{C}\hat{H})_w = i\hbar\, H\, \Lambda C = i\hbar\, \{H, C\}_p$$

Comme H et C sont en involution, il est évident que $\hat{H}$ et $\hat{C}$ commutent.

Ce résultat se généralise de manière évidente au cas d'un nombre de degrés de liberté N quelconque.

2/ Envisageons à présent le cas d'un système hamiltonien intégrable à deux degrés de liberté, associé à un potentiel polynomial et qui possède une seconde constante quartique dans les vitesses. Plusieurs exemples en ont été donnés précédemment [4]. Examinons en particulier :

$$H = \frac{1}{2}(p_x^2 + p_y^2) + 8y^4 + 6x^2y^2 + x^4$$

pour lequel la constante vaut :

$$I = p_x^4 + (24x^2y^2 + 4x^4)p_x^2 - 16x^3y\, p_x p_y + 4x^4 p_y^2 + 4x^8 + 16x^6y^2 + 16x^4y^4$$

Ici, à la différence du cas précédent, le terme $H\, \Lambda^3 I$ n'est pas identiquement nul et vaut $(24)^2 x p_x$.

Pour cette quantification, le système n'est donc pas intégrable. Cependant, il est possible d'ajouter à la constante I un terme constant dans les vitesses, dépendant de $\hbar^2$, de manière à ce que les opérateurs correspondant, toujours dans la quantification de Weyl, commutent.

On vérifie aisément qu'il suffit de prendre :

$$I_1 = I - 12x^2\hbar^2$$

3/ Considérons à présent la famille de potentiels :

$$V = x^{-2/3} y^2 + \frac{3}{4} \lambda x^{4/3}$$

Ces potentiels sont classiquement intégrables lorsque $\lambda = 1, 6, 16$, avec des secondes constantes de mouvement d'ordres respectifs dans les vitesses 3, 4 et 6.

Nous allons ici étudier le cas particulier $\lambda = 1$.

Holt [5] a montré que le système dynamique associé possède, outre l'hamiltonien, une deuxième constante du mouvement cubique dans les vitesses :

$$I = 2p_y^3 + 3p_y p_x^2 + 3p_y(2y^2x^{-2/3} - 3x^{4/3}) + 18p_x yx^{1/3}$$

En calculant comme précédemment le crochet de Moyal de H et de I, on vérifie qu'il se réduit à :

$$\{H,I\}_M = \{H,I\}_P - \frac{\hbar^2}{24} H \overleftrightarrow{\Lambda^3} I$$

Ici, comme dans l'exemple précédent, le terme $H \Lambda^3 K$ n'est pas identiquement nul, il vaut : $+40yx^{-8/3}$.

Pour obtenir des opérateurs quantiques qui commutent, nous sommes donc conduit à perturber les quantités H et K de la manière suivante :

$$\mathcal{H} = H + h(x,y)$$
$$\mathcal{I} = I + e(x,y)\quad p_x + f(x,y)p_y$$

Nous allons à présent calculer h, e, f de manière à satisfaire :

$$\{\mathcal{H},\mathcal{I}\}_M = 0$$

Or :

$$\{\mathcal{H},\mathcal{I}\}_M = p_x^2(-e_x + 3h_y) - p_x p_y(f_x + e_y - 6h_x) + p_y^2(-f_y + 6h_y) + e(V_x + h_x)$$
$$+ f(V_y + h_y) + 18h_x yx^{1/3} + 3h_y\,[2y^2x^{-2/3} - 3x^{4/3}] - \frac{5\hbar^2}{3} yx^{-8/3}$$

Cherchons une solution avec $h_y = 0$.

L'annulation des termes de degré deux dans les impulsions montre que :

$e = 0$

f et h ne dépendent que de x et sont reliés par $f = 6h$

La dernière relation à satisfaire pour le terme constant va fixer la valeur de f : $f = -\frac{5}{12}\frac{\hbar^2}{x^2}$, d'où $h = -\frac{5}{72}\frac{\hbar^2}{x^2}$.

Ce calcul nous montre que pour ce système, il est insuffisant de modifier seulement la constante pour rendre le système intégrable. Il faut aussi ajouter au potentiel une quantité dépendant de $\hbar$ qui redonnera le potentiel initial à la limite classique. La réalisation physique d'un tel système devient alors extrêmement délicate. Hietarinta [6] a montré que, pour les autres valeurs du paramètre pour lesquelles le système classique est intégrable ($\lambda = 6$, $\lambda = 16$), la même perturbation du potentiel conduit aussi à un système quantique intégrable (moyennant une perturbation de la seconde constante).

4/ Examinons à présent l'hamiltonien suivant :

$$H = \frac{1}{2}(p_x^2 + p_y^2) + 2(xy)^{-2/3}$$

Le système dynamique correspondant a été intégré par Fokas [7] et possède une seconde constante cubique dans les vitesses :

$$I = \frac{1}{2}\,(yp_x^2p_y - xp_xp_y^2) - 2(xy)^{-2/3}\,(xp_x - yp_y)$$

Le crochet de Moyal de H et de I a la valeur suivante :

$$\{H,I\}_M = \{H,I\}_p - \frac{\hbar^2}{4}\,H\,\Lambda^3 I$$

$$= -\frac{20\hbar^2}{9}\,(x^{-8/3}y^{-2/3} - x^{-2/3}y^{-8/3})$$

Comme dans le cas précédent, une simple perturbation de I ne suffit pas pour rendre le système quantiquement intégrable, il faut aussi perturber l'hamiltonien H.

En fait, on vérifie que les quantités suivantes :

$$\begin{cases} H_1 = \frac{1}{2}\,(p_x^2 + p_y^2) + (xy)^{-2/3} - \frac{5}{72}\,\hbar^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \\ I_1 = (yp_x^2p_y - xp_xp_y^2) - 2(xy)^{-2/3}(xp_x - yp_y) + \frac{5}{36}\,\hbar^2\left(p_x\,\frac{x}{y^2} - p_y\,\frac{y}{x^2}\right) \end{cases}$$

ont un crochet de Moyal nul et sont donc les extensions naturelles de H et de I qui rendent le système quantique intégrable.

5/ Etude de chaînes de N particules à une dimension :

Nous allons étudier à présent certains modèles de chaînes de particules en interaction sur une droite. Dans la chaîne de Toda [2], les interactions ont lieu uniquement entre proches voisins et sont exponentielles (e^{-q}). Ce système dynamique a été prouvé classiquement intégrable pour un nombre quelconque de particules, lorsque les conditions limites sont périodiques [8]. Plusieurs méthodes ont été utilisées pour cela, dont une qui consiste à calculer explicitement une paire de Lax pour le système. Il en est de même pour la chaîne de Calogéro [3], pour laquelle le potentiel s'écrit :

$$V(x_1,\ldots,x_n) = \sum_{j<k} V(x_j - x_k)$$

$$V(x) = \frac{g^2}{x^2} + \omega^2 x^2$$

ainsi que pour d'autres systèmes similaires étudiés par Perelomov [9].

Tous les systèmes quantiques associés à ces systèmes classiquement intégrables s'avèrent eux aussi intégrables. La paire de Lax associée au système classique permet d'exhiber des constantes du mouvement qui, compte tenu de la forme particulière des potentiels, donnent naissance à des opérateurs commutatifs.

Nous nous proposons d'illustrer ceci sur la chaîne de Toda, suivant la méthode exposée dans [9].

L'hamiltonien de ce problème s'écrit :

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} e^{-(x_{i+1} - x_i)}$$

Les équations du mouvement sont équivalentes à la relation matricielle suivante :

$$\frac{dL}{dt} = [B,L]$$

où :

$$L = \begin{bmatrix} b_1 & a_1 & & 0 & a_N \\ a_1 & & & & \\ & & & & a_{N-1} \\ 0 & & & & \\ a_N & & & a_{N-1} & b_N \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -a_1 & & & -a_N \\ a_1 & & & & \\ & & & & -a_{N-1} \\ & & & & \\ a_N & & & a_{N-1} & 0 \end{bmatrix}$$

avec :

$$a_n = \frac{1}{2} e^{-(x_{n+1} - x_n)/2}$$

$$b_n = \frac{1}{2} p_n$$

Etant donnée la structure de L et de B, on montre que les coefficients du polynôme caractéristique de la matrice L (dét (L-λI)) sont N intégrales du mouvement, indépendantes.

Par exemple, pour N = 3, elles valent :

$$I_1 = p_1 + p_2 + p_3$$

$$I_2 = p_1p_2 + p_2p_3 + p_3p_1 - X_1 - X_2 - X_3 \qquad X_1 = e^{-(x_2 - x_1)}$$

$$I_3 = p_1p_2p_3 - p_1X_2 - p_2X_3 - p_3X_1 \qquad \text{etc.}$$

et de manière générale [9].

$$I_m = \sum p_{i_1} \ldots p_{i_k} (-X_{j_1}) \ldots (-X_{j_e}) \qquad X_m = e^{-(x_{n+1} - x_n)}$$

$$m = 1,2,\ldots,N$$

où $m = k + 2e$ et les indices $i_1,\ldots,i_k$, $j_1,j_1+1,\ldots,j_e,j_{e+1}$ sont tous distincts modulo N.

Remarquons d'abord que I_m définit sans ambiguité un opérateur quantique. Mais, il n'en est pas de même, à priori, pour $[\hat{I}_m,\hat{I}_n]$.

Montrer que ce commutateur est nul et requiert une analyse plus poussée.

On prouve d'abord que :

$$\forall m,\ [\hat{H},\hat{I}_m] = 0$$

Remarquons que l'on a, de manière générale :

$$\left[\sum_{i=1}^{N} \hat{x}_i, \hat{I}_m\right] = i\left(\sum_{i=1}^{N} \frac{\partial}{\partial p_i}\right)\hat{I}_m = i(n-m+1)\hat{I}_{m-1}$$

Si à présent on utilise l'identité de Jacobi, on obtient :

$$i(n-m+1)\,[\hat{H},\hat{I}_{m-1}] = [\hat{H}, [\Sigma\hat{x}_i,\hat{I}_m]]$$
$$= [[\hat{H}, \Sigma\hat{x}_i],\hat{I}_m] - [[\hat{H},\hat{I}_m], \Sigma\hat{x}_i]$$

Or :

$$[[\hat{H},\Sigma\hat{x}_i],\hat{I}_m] = -2i\,[\Sigma_i\, \hat{p}_i,\hat{I}_m]$$

Ce dernier commutateur est nul étant donné l'invariance du système par translations.

Par conséquent, si on prouve que $\hat{I}_N$ est une intégrale du mouvement, il en sera de même de $\hat{I}_m$, $\forall m = 1,2,\ldots,N-1$.

Ecrivons $\hat{I}_N$ de manière à mettre en évidence sa dépendance par rapport à X_1 et (p_1,p_2).

$$\hat{I}_N = (p_1p_2 - X_1)\hat{A}_{12} + B_1\hat{p}_1 + B_2\hat{p}_2 + \hat{C}_{12}$$

où les opérateurs $\hat{A}_{12}$, $\hat{B}_1$, $\hat{B}_2$, $\hat{C}_{12}$ ne dépendent pas de p_1, p_2, X_1.

De même, l'hamiltonien s'écrit :

$$\hat{H} = \frac{1}{2}\hat{p}_2^2 + \hat{X}_1 + \hat{X}_2 + \hat{E}$$

où $\hat{E}$ est de même indépendant de $\hat{X}_1$, $\hat{X}_2$, $\hat{p}_2$.

Dans le calcul de $[\hat{H},\hat{I}_N]$ le seul terme qui résulte de l'action de $\hat{p}_1$, $\hat{p}_2$ sur X_1 est le suivant :

$$i(\hat{B}_2 - \hat{B}_1)\hat{X}_1$$

C'est un opérateur parfaitement défini.

Etant donné que dans le calcul de $[\hat{H},\hat{I}_N]$ n'apparaissent que dés opérateurs bien définis, ce commutateur est nul comme le crochet de Poisson de H et I_N l'est.

Il reste à montrer à présent que :

$$\forall m,k,\ [\hat{I}_m,\hat{I}_k] = 0$$

Or, ce commutateur est une intégrale du mouvement :

$$([\hat{H},[\hat{I}_m,\hat{I}_k]] = [[\hat{H},\hat{I}_m],\hat{I}_k] - [[\hat{H},\hat{I}_k],\hat{I}_m]$$

de degré d'homogénéité totale en p et X, m+k (où chaque X a comme degré 2 et chaque p, 1) invariante par permutation. Mais les coefficients des termes de plus haut degré dans les moments ne sont pas constants. Il en résulte qu'une telle constante est identiquement nulle [9].

Ceci montre que la chaîne de Toda périodique est quantiquement intégrable. Cette démonstration s'étend aussi aux autres chaînes citées plus haut.

CONCLUSION

Nous avons exhibé différents systèmes classiquement intégrables et montré comment cette propriété se conservait pour le problème quantique associé. Dans certains cas, la règle de quantification de Weyl permet directement à partir des constantes du mouvement classique d'obtenir des opérateurs qui commutent (Toda, Calogéro, constantes de degré deux dans les vitesses). Mais, dans d'autres cas, il faut perturber les constantes classiques autres que l'hamiltonien par une quantité en $\hbar$ pour obtenir l'intégrabilité du système quantique associé. Les potentiels :

$$V = x^{-2/3}\, y^2 + x^{4/3}\, \frac{3}{4} \text{ et } V = (xy)^{-2/3}$$

sont un peu particuliers.

En effet, aucune modification de la seconde constante seule ne permet d'obtenir l'intégrabilité du système quantique. Il faut aussi perturber V par un terme en $\hbar^2$.

Même si la réalisation physique d'un tel système est alors plus délicate, ce résultat a l'avantage de montrer que pour tous les systèmes classiquement intégrables que nous connaissons, on peut définir leurs homologues quantiques de façon à préserver l'intégrabilité.

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CHAPITRE 9

ÉTUDE DU CHAOS QUANTIQUE

Depuis quelques années, un intérêt accru est apparu pour l'étude et la description du comportement des systèmes classiques.

Dans le cas d'un système hamiltonien borné de dimension $N > 1$, on rencontre différents types d'orbites. Certaines s'enroulent autour d'un tore de dimension N tandis que d'autres errent de manière stochastique dans des régions contenues sur les surfaces d'énergies (variétés de dimension $2N-1$, d'énergie constante). Les cas les plus rares que l'on rencontre sont les systèmes intégrables où toutes les orbites sont des tores de dimension N, ou à l'opposé les systèmes totalement ergodiques où presque toutes les orbites explorent les surfaces d'énergie. En fait, ce qui caractérise un mouvement chaotique c'est que des orbites initialement proches peuvent s'éloigner complètement et inversement, des orbites distantes initialement peuvent se rapprocher.

A ces différentes classes de mouvements classiques vont correspondre différents états quantiques, états qui sont pour le moment plus ou moins bien connus.

Ce dont on a la meilleure description, c'est du comportement quantique des systèmes intégrables.

Le tore sur lequel s'enroulent les trajectoires forme une base de la quantification, ce qui permet une étude directe. Malheureusement, il n'en est pas de même dans les différents cas où du chaos peut apparaître et si on veut tenter d'étudier les mouvements quantiques correspondant à des systèmes classiques génériques, on est conduit, étant donné la complexité du problème, à choisir les modèles les plus simples possibles.

Les systèmes hamiltoniens indépendants du temps en dimension 2 présentent l'avantage d'être assez simples et en même temps assez riches pour exhiber les différents types de comportement que l'on peut rencontrer.

Parmi eux, ce sont surtout les billards qui ont été considérés pour l'analyse quantique (étude du billard de Sinaï par M. Berry, du stadium par Mc Donald et Kaufman, etc.) [1,2].

On étudie les niveaux d'énergie en fonction des paramètres de déformation du système.

La structure grossière du spectre en énergie ne permet pas de distinguer entre systèmes intégrables et non-intégrables. C'est seulement au niveau des détails fins du spectre que des différences existent. En particulier, pour un système dépendant d'un paramètre, on ne peut pas observer de dégénérescence des niveaux dans le cas non-intégrable (contrairement au cas intégrable où un tel phénomène peut éventuellement se produire). De plus, la distribution d'espacement des niveaux est très différente dans les deux cas. Si le système est intégrable, cette distribution P(S) se comporte comme e^{-S} au voisinage de S = 0 et est donc non nulle (S désigne l'espacement des niveaux). Par contre, il y a répulsion des niveaux dans le cas non-intégrable ($P(S) \to 0$ si $S \to 0$). Cette propriété a été mise en évidence dans de nombreux cas de billards [1].

A côté des systèmes hamiltoniens autonomes en dimension deux, un autre modèle simple peut être fourni par un hamiltonien à un degré de liberté dépendant du temps.

Berry, Balazcs, Tabor, Voros [3] ont étudié le système périodique suivant défini sur une période T par :

$$H(q,p,t) = \frac{p^2}{2\mu\gamma} \qquad (0 < t < \gamma T)$$

$$= \frac{V(q)}{1-\gamma} \qquad (\gamma T < t < T)$$

Cet hamiltonien est successivement soit purement cinétique, soit purement potentiel.

Le mouvement classique se réduit à une application entre les temps nT et (n+1)T, définie par :

$$\begin{cases} q_{n+1} = q_n + p_n \dfrac{T}{\mu} \\ p_{n+1} = p_n - T \dfrac{\partial V}{\partial q}(q_{n+1}) \end{cases} \qquad (1)$$

Pour $V(q) = \frac{1}{2} Aq^4$, le système est non-intégrable. Berry *et al.* ont regardé l'évolution de la fonction de distribution au cours du temps. Les déformations laissent apparaître des volutes et des vrilles qui sont des convolutions compliquées liées aux points fixes de l'application *(1)*. Si on attend suffisamment longtemps, cette structure chaotique se développe et peut apparaître à des échelles arbitrairement petites. Quantiquement, on ne peut pas donner de sens à des structures plus petites que $\hbar^N$ où N est la dimension du système. En conséquence, à long terme, la correspondance classique-quantique est brisée et seuls les effets quantiques dominent. Ce phénomène s'observe sur l'évolution de la transformée de Wigner de la matrice densité (dans l'espace des phases) ainsi que sur l'étalement de cette distribution au cours du temps (Korsch et Berry [4]). Contrairement à ce qui se produit pour la majorité des systèmes intégrables (où l'effet quantique est de surajouter des détails à la structure grossière du comportement classique), les phénomènes quantiques ont ici pour effet de lisser la structure fine, à long terme.

Dans la même lignée, le modèle du rotateur pulsé périodiquement, introduit par Casati, Chirikov, Izraelev et Ford [5], présente l'avantage de pouvoir être étudié analytiquement, au moins en partie, et de donner lieu classiquement à un mouvement chaotique.

L'hamiltonien, pour ce problème, a la forme suivante :

$$H = \frac{I^2}{2} + V(\theta) \sum_{n=0}^{\infty} \delta(t - nT)$$

avec $V(\theta) = k \cos\theta$ *(1)*.

Comme précédemment, le mouvement classique se ramène à une application entre les temps nT et (n+1)T.

La quantité intéressante à considérer est l'évolution de l'énergie moyenne au cours du temps. Dans le mouvement classique, on observe une croissance aléatoire, linéaire en moyenne, de l'énergie. Le mouvement quantique est plus compliqué et son étude a fait entre autres l'objet de plusieurs travaux récents, dont la publication qui vient à la suite.

Nous y étudions l'influence de l'amplitude de la perturbation k sur le comportement de l'énergie du système.

Par des études numériques, nous mettons en évidence le fait que pour des petites valeurs de l'amplitude, cette énergie semble récurrente dans l'intervalle de temps considéré (même lorsque l'on est dans un cas de résonance où le rapport de la fréquence des perturbations à la fréquence propre du système est égal au rapport de deux entiers naturels suffisamment grands). Pour des valeurs plus grandes de l'amplitude, l'énergie semble par contre croître avec le temps, soit en t^2 dans le cas résonant ou plus lentement dans le cas non-résonant.

Une application unitaire définit l'évolution du système pendant une période du temps. Nous nous sommes aussi intéressés à l'étude du spectre de cette application. Pour cela, nous avons fixé le rapport des fréquences égal au nombre d'or et constaté l'absence de lois d'échelle pour différentes approximations de ce nombre.

L'interprétation de ces résultats reste pour le moment très qualitative et nécessiterait des études plus poussées. Ceci peut procéder par l'intermédiaire d'analyses plus fines du modèle du rotateur pulsé ou par l'étude de l'influence de perturbations dépendantes du temps sur d'autres hamiltoniens quantiques.

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THE PERIODICALLY KICKED ROTATOR : RECURRENCE AND/OR ENERGY GROWTH

B. Dorizzi, B. Grammaticos
CNET, Département de Mathématiques
92 Issy les Moulineaux, France

Y. Pomeau
SPh.T, CEN Saclay
91191, Gif-sur-Yvette Cedex
France

and
Laboratoire de Physique de l'ENS
24 rue Lhomond, 75005 PARIS
France

ABSTRACT

We explore the properties of the quantum kicked rotator. Its classical equivalent being the standard map. Its behavior, as found by computer studies, depends very much on the strength of the external forcing.At low strength it is seemingly recurrent in the sense of Hogg and Huberman. However its energy increases with time at large forcings. For quantum systems, a unitary map defines the evolution over one period of time. The spectrum of this map in an infinite space does not seem to be a well defined concept when the frequencies of the external forcing and of the unperturbed system are in the ratio of the golden mean.

Since the work of Casati et al [1] , several works have been devoted to the understanding of time dependent quantum systems beyond the usual perturbation expansions. One of their motivations was to study whether the complex dynamics of classical non-integrable generic systems has any counterpart in quantum systems. A most popular model for this study is the rigid rotator submitted to periodic kicks. Its time dependent Schrödinger equation (in dimensionless units) is :

$$i \frac{\partial \psi}{\partial t} = - \frac{1}{2} \frac{\partial^2 \psi}{\partial \theta^2} + k\, V(\theta)\, S(t)\, \psi$$

where the wavefunction $\psi(\theta)$ is smooth and 2π periodic ($\psi(\theta + 2\pi) = \psi(\theta)$). Here k measures the strength of the time dependent perturbation, $V(\theta)$ the position dependence of the potential (most often $V(\theta) = \cos\theta$) and $S(t)$ is a sum of kicks :

$$S(t) = \sum_{N=-\infty}^{+\infty} \delta(t - NT)$$

with T being the time lag between two consecutive kicks.

The classical counterpart of this model described by the so-called "standard mapping" [2] , is a well known example of chaotic behaviour. The energy of the classical kicked rotator grows with time in a chaotic way following, in the mean, a diffusion law $E \sim t$ provided the perturbation is sufficiently large.

In the quantum case several different behaviours have been isolated. The question of the convenient initial condition for observing this kind of behaviour is discussed below. For the moment it is enough to say that we consider everywhere the energy growth for initial condition of finite energy.

In the work of Casati et al [1] , extended later by Izrailev and Shepelyansky [3] the phenomenon of quantum resonance has been observed. Whenever the period of the kicks T is a rational multiple of 4π the energy grows quadratically with time : $E \sim t^2$. Whenever the resonance condition is not satisfied it is tempting to call the case non-resonant. However, various behaviours of the energy are still possible and one must be particularly careful. Working with an irrational $T/4\pi$ and with moderate perturbation, Chirikov et al [4] have found numerically a saturation phenomenon. Namely the energy starts growing with time but rapidly the

growth slows down and proceedswith a much smaller rate if there is any growth at all .On the other hand, Hogg and Huberman have introduced the so-called recurrence property [5] . They have shown that, in what they called the "non resonant" case, the energy must come back arbitrarily close to its initial value during the evolution of the system. However what Hogg and Huberman called "non resonant" may turn out to be very restrictive (beyond excluding what is called resonant by Izrailev and Shepelyansky). The evolution from time t to time t+T defines a unitary map U on the L^2-space of wave functions on $[0,2\pi[$. The system is non resonant, on the sense of Hogg-Huberman, if a complete and discrete set of eigenfunctions on L^2 exists such that

$$U\phi_n = e^{i\alpha_n}\phi_n \quad , \quad n \in \mathbb{N}^{\alpha} \ , \ \alpha \in \mathbb{R}$$

For instance it is not clear that such a property holds for the Hamiltonian of the kicked rotator for some irrational $T/4\pi$. Although the resonant case of Izrailev-Shepelyansky is excluded because the spectrum of U is continuous if $T/4\pi$ is rational . The numerical computations of Chirikov et al for irrational $T/4\pi$ show an energy growth, as we mentioned above. Estimates of the recurrence time [6] in some realistic cases have yielded absolutely large times, which allows one to conclude that the recurrence, although present is quite sensitive to the magnitude of the perturbation.

Attempts have been made [7] to relate the properties of time-dependent quantum systems, such as the kicked rotator, to the problem of localization in one-dimensional quasi-periodic potentials. The role of the quasi-periodic potential is, very roughly speaking, played by the phase factor $\beta_m = e^{im^2T/2}$, $m \in Z$ being the site of the "1-d lattice", and $T/4\pi$ irrational. Using We yl's theorem [8] it is easy to show that $\langle\beta_m\rangle = 0$, the average being taken over the values of m. Contrary to quasi-periodic potentials , the pair correlations $\langle\beta^*_m\beta_{m+p}\rangle = 0$ vanishes at fixed p. However the four-point correlations as $\langle\beta^*_m\beta_{m+p+q}\beta^*_{m+q}\beta_{m+p}\rangle$ do not decay at all for large p's and q's implying that the fluctuations of β do not have the cluster property, as the potential fluctuations in the quasi-periodic case.

In the present paper we reconsider the problem of the kicked rotator from various points of view. As we will present several numerical results

we will start by checking on details the exact predictions that can be made on the resonant case. Apart from the intrinsic interest of such a study this will serve as a stringent test for our numerics . We then will analyse the non-resonant case and show how the pictures of Chirikov et al ón the one hand and Hogg-Huberman on the other hand can be reconciled : for k sufficently large, the energy increases with large fluctuations while at small k the energy fluctuates without any apparent growth almost exactly as in the recurrence picture. Motivated by recent investigations on the localization problem we have examined the spectrum of eigenvalues of V by approaching an irrational $T/4\pi$ (here the golden mean $(\sqrt{5}-1)/2$) by rational approximations of the Legendre continued fractions. No scaling properties (as in the localization case) were found, which might be an indication that the quasi-energy spectrum lacks any regularity. In the conclusion of this paper we recapitulate our results and present physical systems whose behaviour could be described by the dynamics of the kicked rotator.

The model

Let us start with the Hamiltonian of the periodically kicked rotator :

$$i \frac{\partial \psi}{\partial t} = - \frac{1}{2} \frac{\partial^2 \psi}{\partial \theta^2} + k\, V(\theta)\, S(t)\, \psi \quad . \tag{1}$$

Here $S(t) = \sum_{N=-\infty}^{+\infty} \delta(t - NT)$. Introducing the dimensionless time $\tau = \frac{t}{T}$ we obtain

$$\frac{\partial \psi}{\partial \tau} = i \frac{T}{2} \frac{\partial^2 \psi}{\partial \theta^2} - ik\, V(\theta) \sum_{N=-\infty}^{+\infty} \delta(\tau - N)\, \psi \quad . \tag{2}$$

In order to integrate equation (2) we distinguish a two step evolution, between successive kicks and at the kicks. The integration between kicks can be simplified if one expands ψ on the basis of the wavefunctions of the the static rotator : $\phi_n = \langle \theta | n \rangle = e^{im\theta}/\sqrt{2\pi}$:

$$\psi(\theta, t) = \sum_{n=-\infty}^{\infty} a_n(t) \frac{e^{in\theta}}{\sqrt{2\pi}} \tag{3}$$

Introducing $V_{nm} = \langle n | V(\theta) | m \rangle$ we obtain

$$\frac{\partial a_n}{\partial \tau} = - i \frac{T}{2} n^2 a_n - i k\, S(\tau) \sum_{m=-\infty}^{+\infty} V_{nm}\, a_m \tag{4}$$

By writting further $a_n = e^{-i\tau Tn^2/2} b_n$ we have

$$\frac{\partial b_n}{\partial \tau} = -i\, k\, S(\tau) \sum_{m=-\infty}^{+\infty} W_{nm}\, b_m \qquad (5)$$

with $W_{nm} = V_{nm} e^{i(n^2 - m^2)\frac{\tau T}{2}}$.

Between kicks $S(t) = 0$ and we have simply $\frac{\partial b_n}{\partial \tau} = 0$. Thus $b_n^-(N) = b_n^+(N-1)$ where N denotes the kick and +(-) the moment just after (before) the kick. At the kick itself eq. (5) can be integrated (over an infinitesimal interval which encompasses the kick) to give :

$$b_n^+(N) - b_n^-(N) = -i\frac{k}{2}\left(\sum_m W_{nm}(N)\, b_m^+(N) + \sum_m W_{nm}(N)\, b_m^-(N)\right) \qquad (6)$$

or equivalently :

$$b_n^+(N) + i\frac{k}{2}\sum_m W_{nm}(N)\, b_m^+(N) = b_n^+(N-1) - i\frac{k}{2}\sum_m W_{nm}(N)\, b_m^+(N-1) \qquad (7)$$

We have thus obtained a simple mapping which relates the wave-function components at two successive kicks. Dropping the supercript + and with obvious matrix notations the mapping (7) can be written as

$$b(N) = \frac{I - \frac{ik}{2} W(N)}{I + \frac{ik}{2} W(N)}\, b(N-1) \qquad (8)$$

We remark thus that we have an explicitly unitary scheme for the iteration of the wavefunction components. The matrix inversion involved can easily (and accurately) be performed using Gauss' algorithm and can be particularly simplified if one chooses the form of $V(\theta)$ so as to cancel most of the off-diagonal matrix elements. The usual choice $V(\theta) = \cos\theta$ is not the best one from this point of view, leading to complicated matrix elements. A far simpler choice is $V(\theta) = \frac{2}{k}\,\mathrm{arctg}(\frac{k}{2}\cos\theta)$ (which was also proposed by Prange et al) and which leads to matrix elements for the kick action of the form $V_{nm} = \delta_{n,m\pm1}$. Incidentally, for small k, the two potentials are equivalen

As far as the numerical implementation of the iteration scheme is concerned, we must stress the fact that unitarity is not a crucial test of the accuracy of our method, being explicitly built into the algorithm. It is in fact conserved up to the accuracy of the computer arithmetic. A far more

important constraint is due to the spreading of the wave function which requires the presence of several hundred terms in the expansion (3) of ψ . The space needed increases very rapidly with k, and at some cases up to 10000 components were necessary (although due to the symmetry between a_n and a_{-n} the working space dimensions can be reduced by a factor of two).

Study of the resonant case

In the original study of Casati et al. the case of the fundamental quantum resonance was studied in detail. It corresponds to a period, for the kicks, which is an integer multiple of 4π : i.e. $T = 4\pi m$. One remarks easily that in this case the phase $e^{-iTn^2/2}$ which is the evolution of the amplitudes a_n between kicks is equal to 1. Casati et al have proven that in this case the energy grows quadratically with time $E \sim \tau^2$.

Izrailev and Shepelyansky have extended this result to the case of a resonant period of the form $T = 4\pi \frac{p}{q}$ with integer p and q. In this case the phases recover the value 1 only after q kicks but this leads again to a quadratic energy growth. $E = \eta\tau^2 + O(\tau)$. They have also presented a numerical evaluation of the growth coefficient η , as well as asymptotic estimates of the latter, which however were not verified numerically in the small k limit.

In our study we have examined in detail the case $\frac{p}{q} = \frac{8}{13}$. In figures 1,2,3 we present three typical pictures for the behaviour of the energy as a function of time. The values are respectively k = .8, 2.0, 4.8. One notices that for the smaller value of k the energy growth is imperceptible at the beginning. If we have stopped our evolution at $\tau \sim 100$-200 we would have obtained a picture of energy recurrence "à la Hogg-Huberman".On the contrary with increasing k the recurrence behaviour disappears and for the larger values of k we obtain a quadratic growth with practically no wiggles. This behaviour reflects the crucial dependence of the energy on k, a feature which we will encounter again in the non resonant case.

A systematic study of the energy growth as a function of k has allowed us to verify the asymptotic estimates of η , proposed by Izrailev and Shepelvansky. For $k \ll q$ they have found that $\eta \sim k^{2q}$, while for $k \gg q$ $\eta \sim k$ (Actually the aforementioned authors have obtained $\eta \sim k^2$ in the latter case, using the potential $V = k \cos\theta$. The value $\eta \sim k$ is associated to the potential $V = 2 \operatorname{arctg}(\frac{k}{2}\cos\theta)$ used in the present work).

In figure 4 we present the results for η as a function of k, over a large range of values of the latter, which allows us to reach the two asymptotic limits, materialized on the figure by the two straight-line asymptotes. One remarks readily that the agreement is quite satisfactory for large as well as for small k. i.e. for behaviours of η ranging from $\eta \underset{k\to\infty}{\sim} k$ to $\eta \underset{k\to 0}{\sim} k^{2q}$.

One remark is in order at this point. The term quantum resonance may be somewhat misleading as it could be interpreted as a phenomenon present only in the quantum case. This is however not true. Quadratic energy growth can be present in the classical case as well. However in the latter case, it corresponds to highly unstable trajectories, and thus very difficult to study numerically.

The non resonant case.

As explained before, this is the case where the period of the kicks is not commensurate with the natural period of the system that is $\frac{T}{4\pi}$ is irrational. Indeed this is the most general case. However as we are interested in the limit $t \to \infty$, one could imagine that the energy $E(t,\frac{T}{4\pi})$ with given initial conditions behaves as $\varphi(t)$ for $t \to \infty$ and almost all $\frac{T}{4\pi}$, even though the average of $E(t,\frac{T}{4\pi})$, say $\tilde{\varphi}(t)$, over a set of values of $\frac{T}{4\pi}$ of finite measure at a given time is such that $\tilde{\varphi}(t) \underset{t\to\infty}{=} \varphi(t)$.

Below we shall give the numerical behavior we did find for the energy as a function of time, then we shall try to interpret our results.

Numerical results : energy growth

Following a widely spread habit, we have chosen for $\frac{T}{4\pi}$ the golden mean $\frac{\sqrt{5}-1}{2}$. This irrational number can be represented as $\lim_{n\to\infty} \frac{P_{n-1}}{P_n}$, $\{P_n\}$ being the Fibonacci with $P_o=P_1=1$ and $P_{n+1} = P_n+P_{n-1}$. From the point of view of continued fraction expansion, the golden mean is -in a sense- the "most irrational" number. We then computed the energy $E(t, \frac{P_{n-1}}{P_n}; k)$ with the ground state $\psi(\theta) = \frac{1}{\sqrt{2\pi}}$ as initial condition in eq. (1). For a given value of 1 and given t_o, we observed that the curve $E(t, \frac{P_{n-1}}{P_n}; k)$ has a rather well defined limit as n becomes large for any t in the interval $[0,t_o]$. We took this limit value as the value of $E(t,\frac{T}{4\pi}; k)$ where $\frac{T}{4\pi}$ is the golden mean.

Using the limiting process in the approximation of the golden mean (which we recall depends on the time interval t_o for given k) we have performed several calculations at selected values of k. For low k, typically k smaller than 10, the energy fluctuates as a function of time without any secular trend over a time interval at 1000 kicks (fig.5). A similar trend was observed in the case of low order resonances. So at low k's a recurrence behaviour "à la Hogg-Huberman" of the energy is observed.

At larger k the energy increases first very rapidly, roughly linearly, the initial slope increasing with k. However this phenomenon persists only for about 50 kicks, and then the energy growth saturates. For k's of the order of ~100 a small energy growth accompanied by large fluctuations is observed after the initial burst (fig.6). At still larger values of k the relative importance of the fluctuations becomes much larger, and the secular trend of the energy towards growth, although present, is not always easily discernible (fig.7). A qualitative similar behaviour was observed by Chirikov et al. [4].

At this point we must remark that it is quite difficult to investigate numerically, with a satisfactory degree of accuracy, a law for the final growth of the energy, and it is also extremely difficult to know if there exists a sharp transition from a low-k regime, without energy growth, to a high-k regime, where the energy grows with time. Let us assume, for instance that the energy behaves as :

$$E(t, \frac{T}{4\pi} ; k) \underset{t\to\infty}{\sim} t^{\mu} D(k) \quad , \quad \mu \sim 1$$

The function D(k), if not zero near k = 0, should be transcendentally small for $k \to 0+$, as for example, $e^{-k_o^2/k^2}$, because this kind of energy growth is out of reach of expansions in powers of k. Numerically one cannot distinguish between a function growing from zero as $e^{-k_o^2/k^2}$ and a function vanishing exactly between zero and, say, $k_1 \sim 0.1\ k_o$ followed by a smooth behaviour after k_1.

To test the sensitivity of our results to the peculiarities of the model, we have performed the following "experiment". In the numerical algorithm, the unperturbed Hamiltonian occurs only through the phase factors : $\exp(im^2T/2)$. We have added random phases ε_m $(0 < \varepsilon_m < 1)$ to $\frac{m^2T}{2}$, destroying thus the resonance condition for rational $T/4\pi$. If one

establishes the analogy with the 1-D localisation problem, this small amount of randomness is a strong perturbation indeed, as it leads to exponential localization. In the present case the addition of random ε_m's changes dramatically the pattern of energy growth in the resonant case. However the general trends are the same as in the non resonant case : no energy increase for low k, and increase of energy with large fluctuations for larger values of k . (fig.8).

Thus our results agree with what was previously known. Nevertheless we have shown that there exists a very strong sensitivity with respect to the value of k (a point that has not been raised till now). The various behaviours, recurrence or growth-saturation, that were presented before, are merely manifestations of the dependence of energy on the perturbation magnitude k.

Toward an interpretation of our results.

As explained before, it is still difficult, if not impossible to interpret our results in the light of available theories. These theories make some assumptions for the quasi energy spectrum, that is for the spectrum of the one step unitary operator [9].

For the present model this spectrum is known in two limit cases. First of all if k = 0 (no kicks), this spectrum is simply deduced from the set of energy levels $\{E_m\}$ of the free rotator. The quasienergies are $(m^2T/4\pi)$, where, as usual, (x) is the fractional part of x. From Weyl's theorem, the numbers $(m^2T/4\pi)$ are uniformly distributed in $[0,1[$ as m varies in $\mathbb{N}$ for almost all $T/4\pi$. The precise diophantine condition on $(T/4\pi)$ for which this is true poses a seemingly non trivial problem, but we can be sure that $(m^2T/4\pi)$ is not uniformly distributed if $T/4\pi$ is rational.

If $k \neq 0$, the quasi-energies are known in the resonant case. They fill continuous bands. We have already said that a possible connection has been claimed between the present problem and the quantum localization in one dimensional quasi-periodic potentials. In this last case, it is now understood that in some cases the energy spectrum has a Cantor like structure and this appears step by step when the irrational ratio of the two spatial frequencies is approached by rational approximations of increasing accuracy [10].

We sought similar regularities in the quasi energy spectrum by approximating the golden mean for $T/4\pi$ by the ratios of successive Fibonacci numbers. Practically, we took 2/3, 3/5, 5/8 and 8/13.

As shown in Figure (9), the band structure of the quasi-energy spectrum at these resonances have no apparent ressemblance. In particular the band splitting leading (eventually) to a Cantor set does not show up.

This could be explained as follows. The bandwidth of each band is approximately of order $(\frac{k}{q})^q$ for large denominators q in the rational approximation of an irrational $(T/4\pi)$.

When one considers higher and higher approximation of the golden mean with $q_n \sim \frac{1}{2}(\frac{1+\sqrt{5}}{2})^n$, the width of the bands decreases much faster than in a geometric fashion. This excludes a Cantor structure with a more or less constant reduction ratio at each step, as the one occuring in the localization problem. Moreover, as the bands have a thickness $\Delta_n \sim (k/q_n)^{q_n}$, they manifest in a sense the behavior of the system over time scales of order Δ_n^{-1}. For such times, the band structure at order (n-1) is destroyed, because $\Delta_{n-1}\,\Delta_n^{-1} \gg 1$. This is because the bands of the quasi-energies of the matrix U^T (T positive integer) have a thickness equal to T times the thickness of those of U.

Thus, our numerical results and the previous considerations indicate that no point wise limit of the quasi-energy spectrum exists as n increases. In some sense, at each arithmetic approximation of $(T/4\pi)$, this spectrum shows a new pattern, almost independent of the spectra at lower approximations. Indeed, this leads one to think that, as n increases, there are statistical rules for generating these quasi-energy spectra. But these rules are, for the moment, unknown, and out of reach of our computational power.

The lack of regularity of the time dependence of the energy for the non resonant case should be explained along the same lines : at time t, the system is sensitive to the structure of the quasi-energy spectrum at a rational approximation of $(T/4\pi)$ with a denominator of order t^{-1}, and this structure is very sensitive to this rational approximation.

Nevertheless, for the moment, all these considerations are very qualitative. We hope to make them more definite in the future.

Conclusion

Our numerical study clearly indicates the great importance of the strength of the perturbing force in these time dependent quantum problems : at low intensities, and except for resonance with small denominators, the energy seems to be recurrent, although for large intensities this has a tendency to grow with time, either as t^2 in the resonant case or more slowly in the non resonant case.

It is of interest to speculate about the applicability of this kind of theory to experiments. Perhaps the most natural idea for this is to think at atomic or molecular physics, the external force being due to an electromagnetic wave. Unfortunately, the dimensionalizing parameter for the strength of this wave will be the electric field in a Bohr atom ($\sim$ 1 volt over a few Angströms), a huge field when compared to electric fields in electromagnetic waves. And it should be difficult to reach experimentally a domain where non perturbative effects (with respect to the amplitude of the external field) were important. Nevertheless the internal electric field is much lower than 1V/1Å in the so called Rydberg states (hydrogenic state with a huge principal quantum number), and this should be considered as a candidate for such non perturbative effects.

One might try also to apply this kind of theory to macroscopic quantum states. To stay as close as possible to the quantum rotator, one should think at a superfluid loop in its ground state and accelerated periodically parallel to its own plane. Indeed, this could be compared with the quantum rotator if no vortex were generated, so that the only excitation of the system were quanta of circulation around the loop.

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Figure Captions

Fig.1. Energy, E, for the rotator as a function of time (number of kicks) nt, for $T/4\pi = 8/13$ and $k = .8$.

Fig.2. Same as in fig. 1 for $k = 2$.

Fig.3. Same as in fig. 1 for $k = 4.8$.

Fig.4. Factor η of the quadratic energy growth ($E \sim \eta t^2$) as a function of k (in logarithmic scales) for $T/4\pi = 8/13$.

Fig.5. Energy as a function of time for $T/4\pi$ a rational approximation of the golden mean (here $T/4\pi = 17711/28657$) and $k = 4$.

Fig.6. Same as in fig. 5 for $k = 80$.

Fig.7. Same as in fig. 5 for $k = 160$.

Fig.8. Energy as a function of time for $T/4\pi$ and $k = 8$. after the addition of small random shifts to the unperturbed spectrum.

Fig.9. Quasi-energy spectrum, in units of π , for reasons small rational approximations to the golden mean. For $\frac{p}{q} = 5/8$ all the bands are doubly degenerate. For $\frac{p}{q} = 8/13$ one band is doubly degenerate.

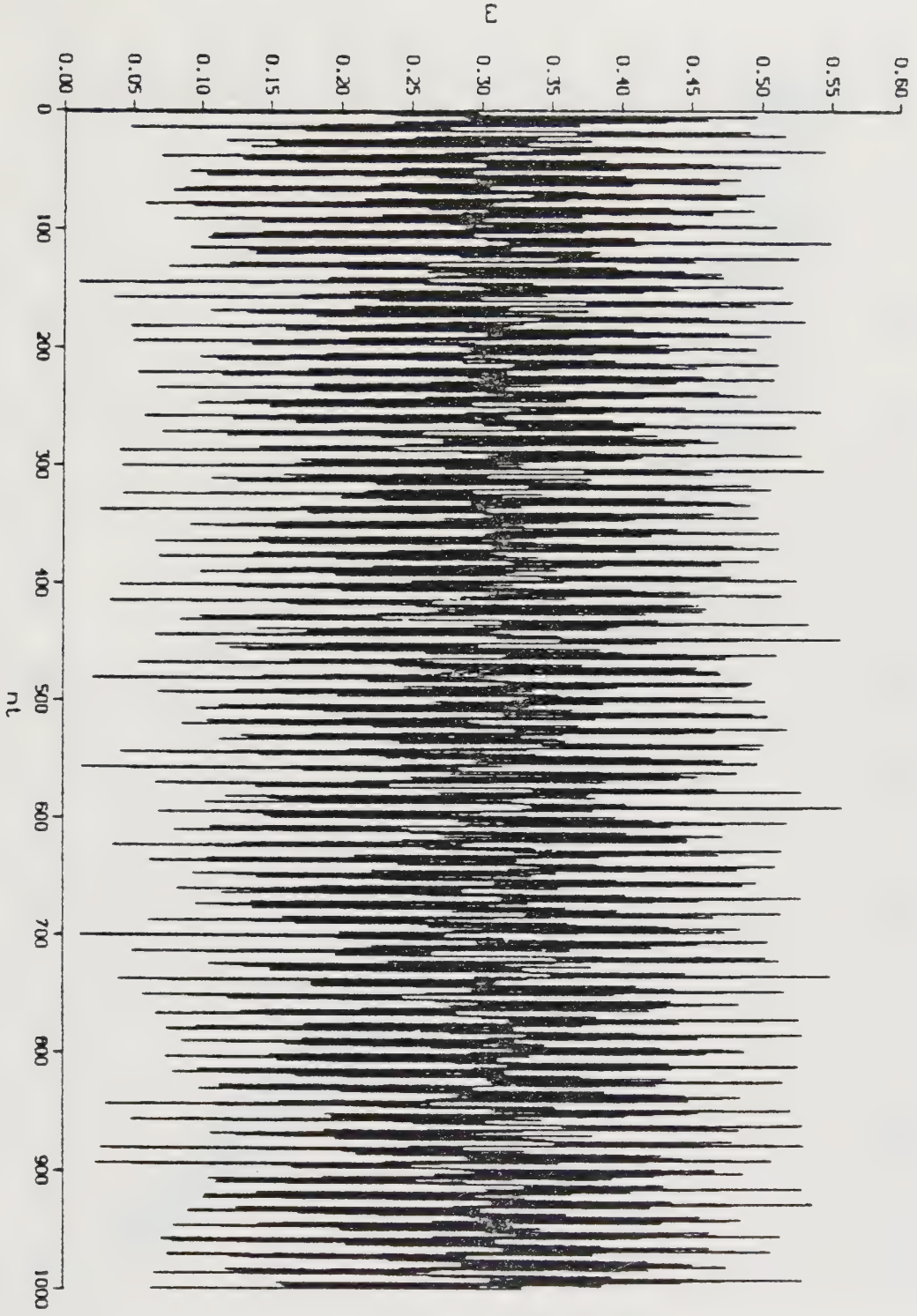
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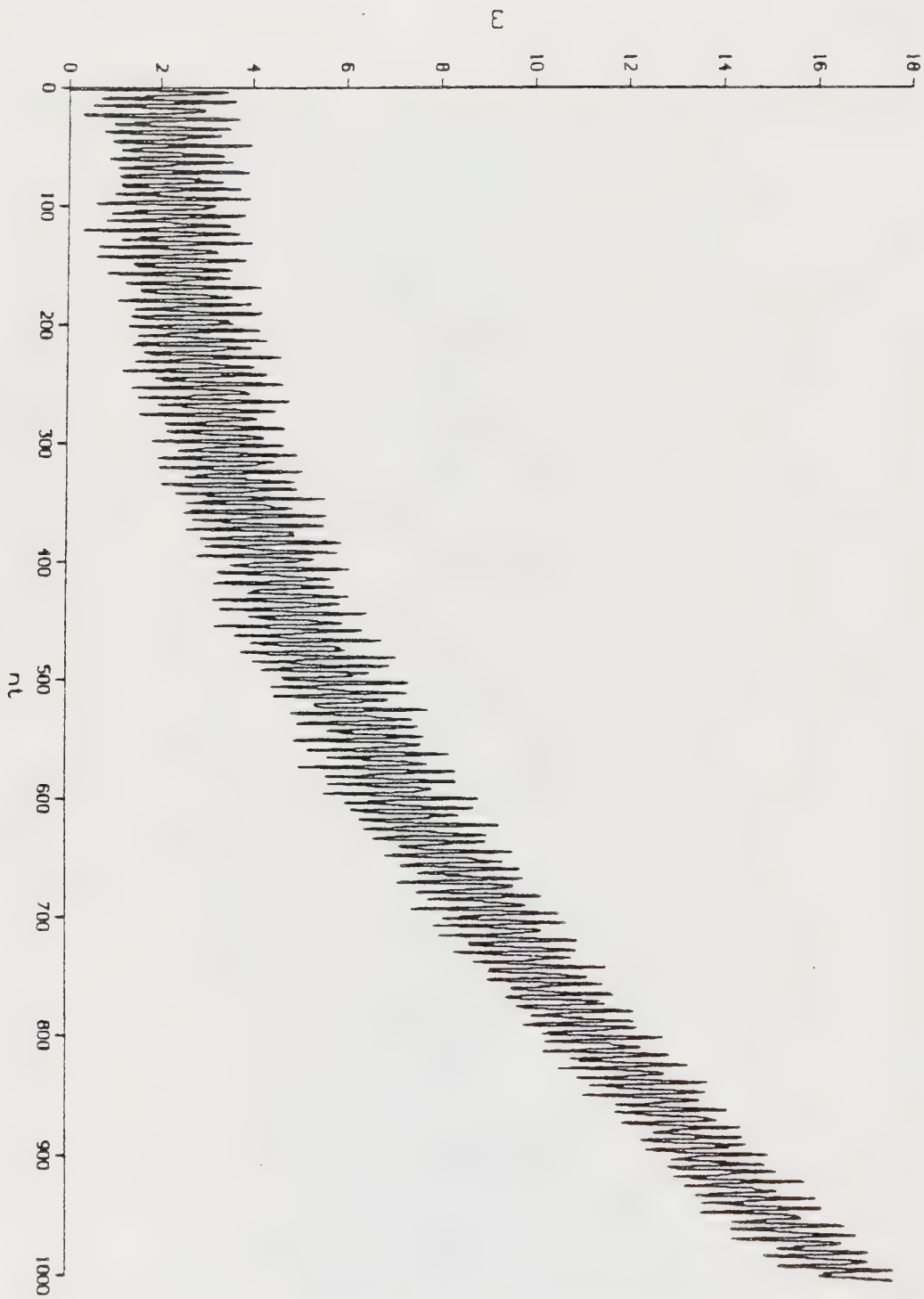
Figure 1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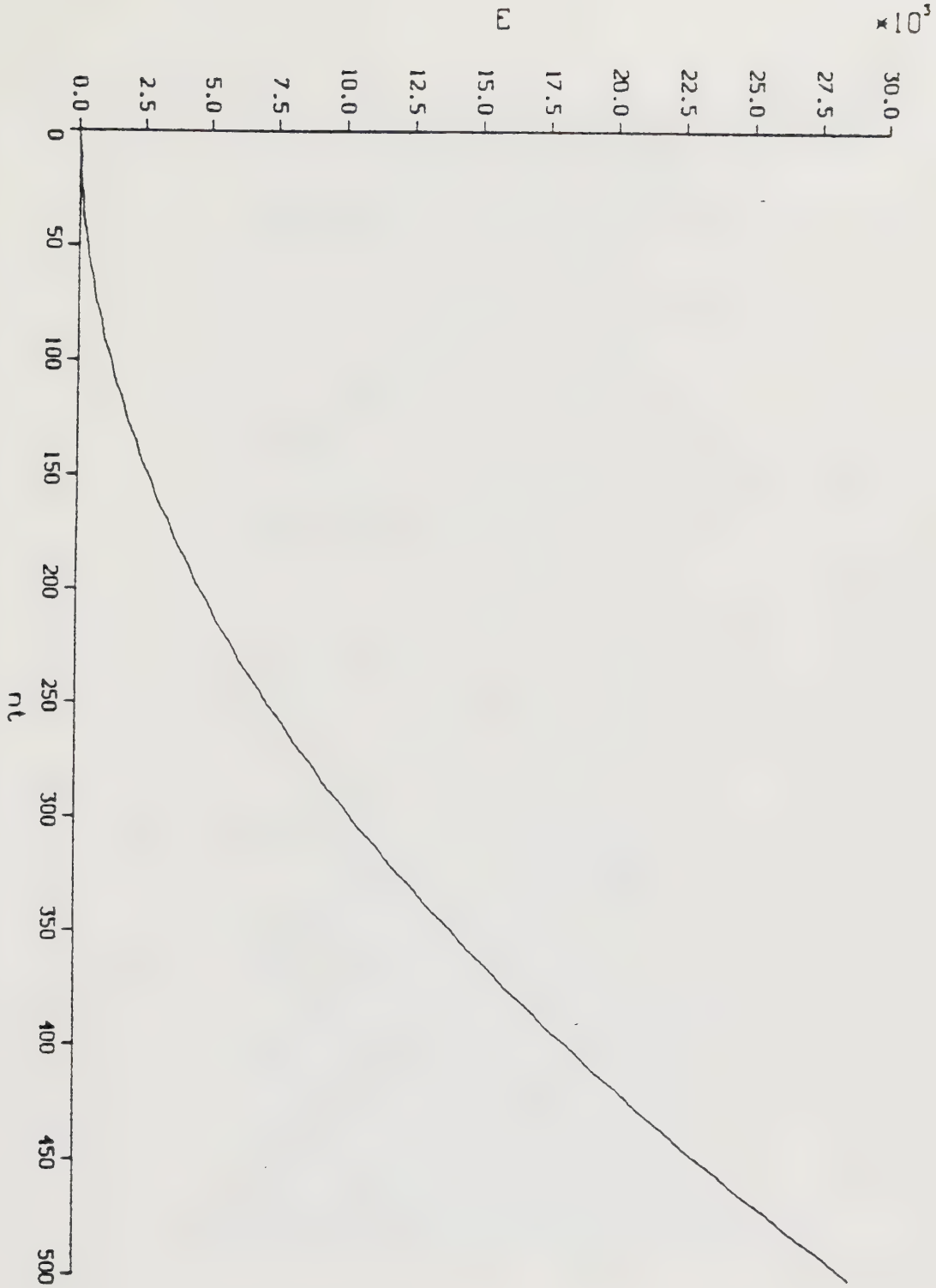
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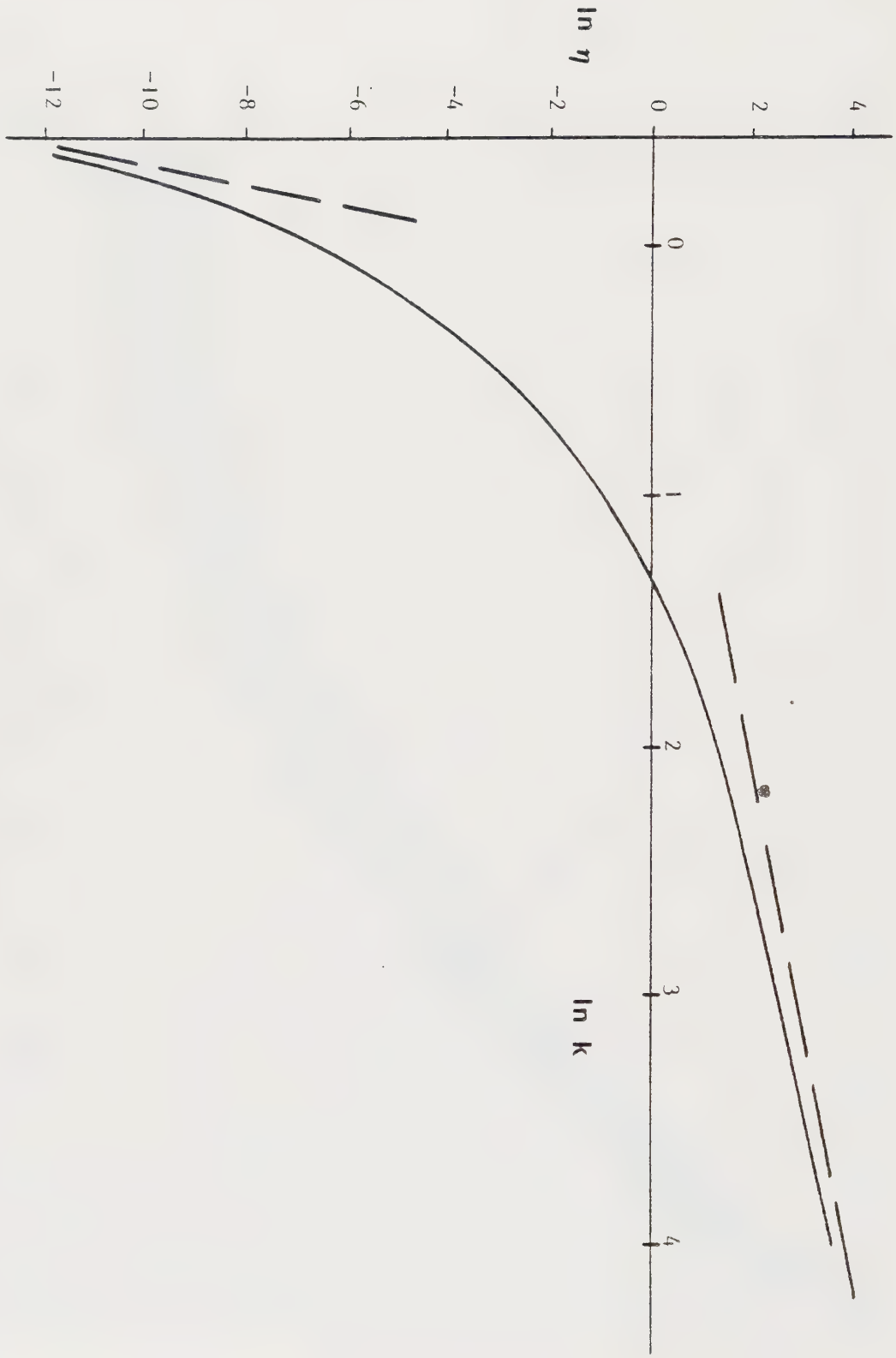

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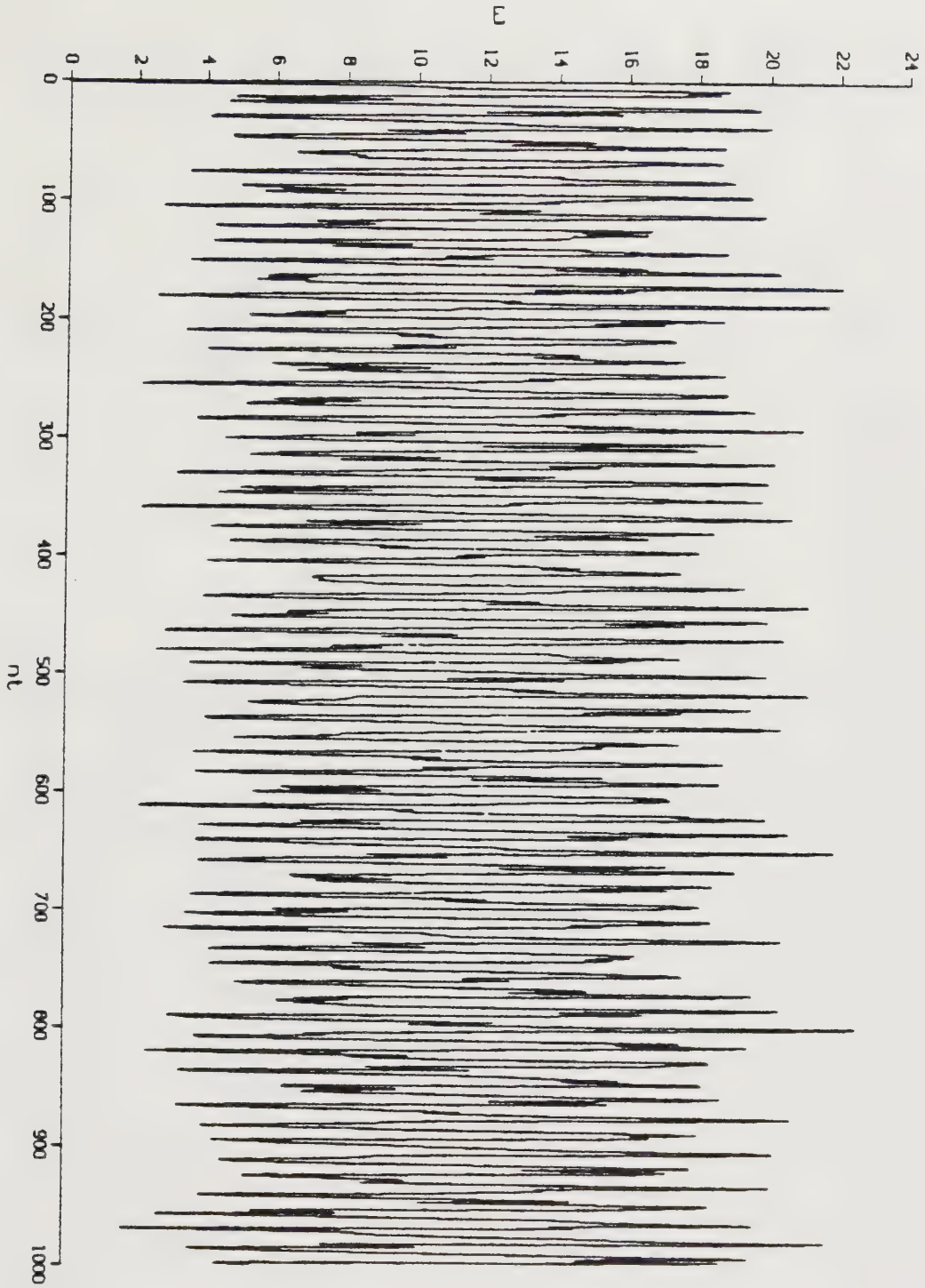
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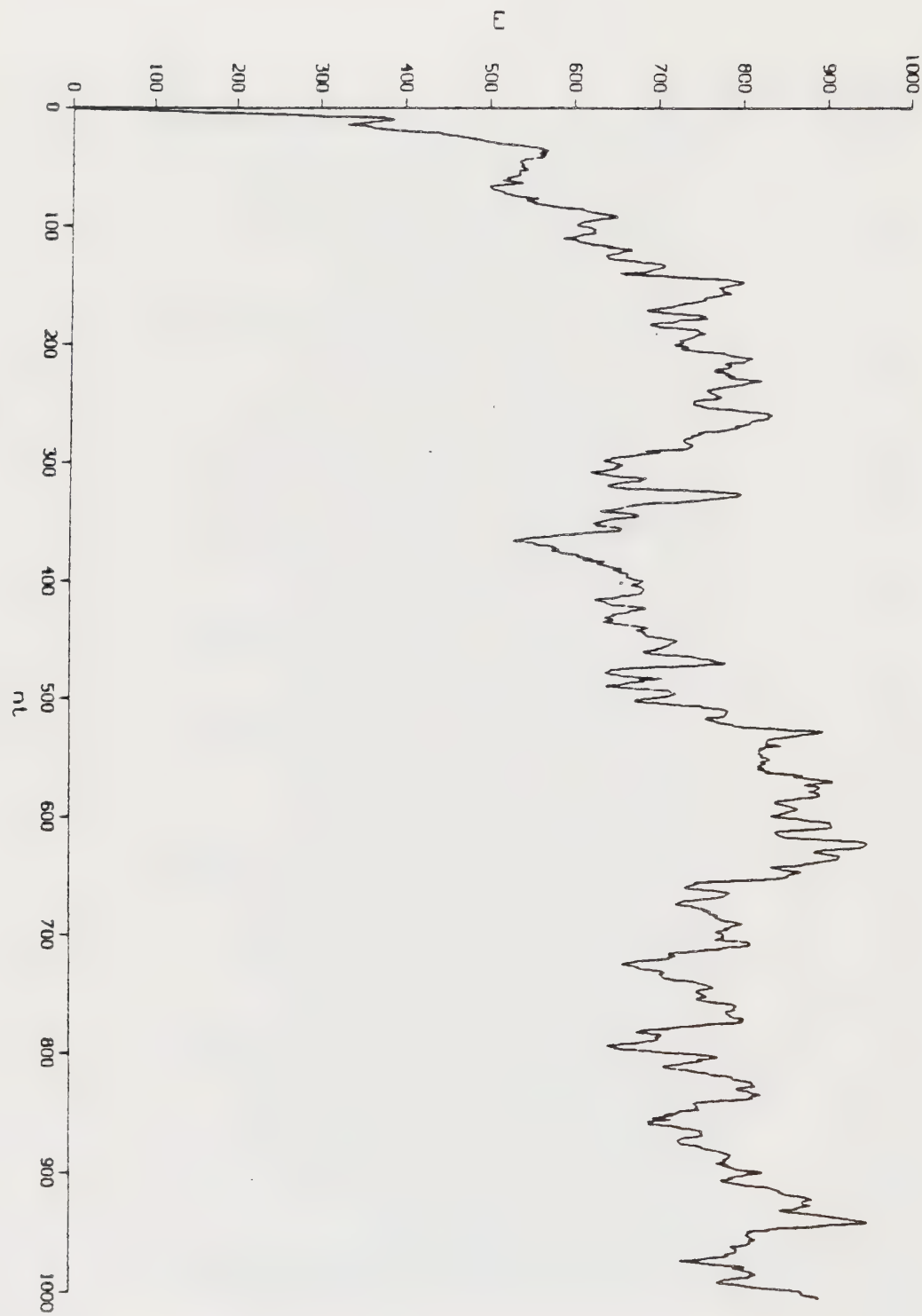
Figure 5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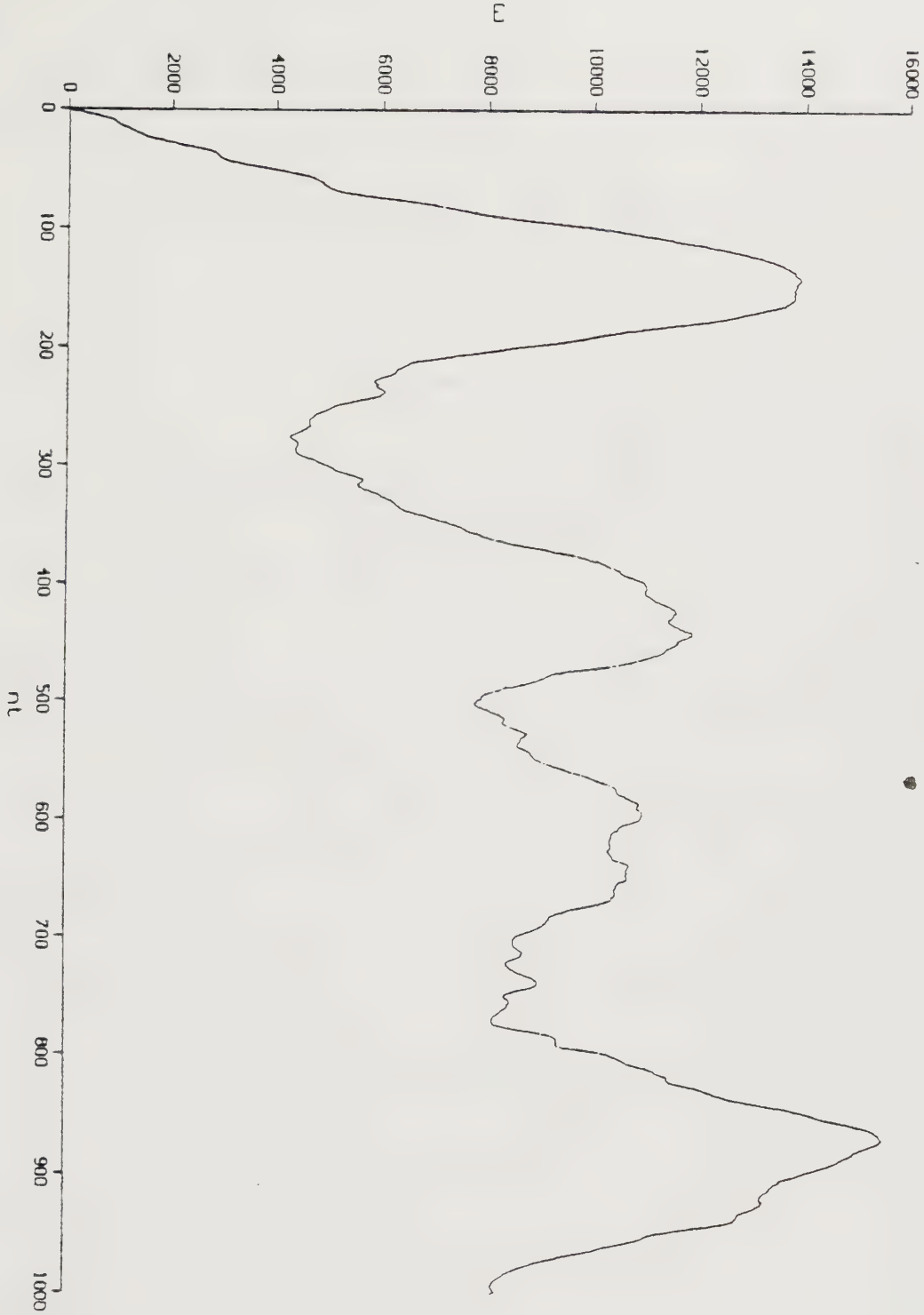
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Figure 7

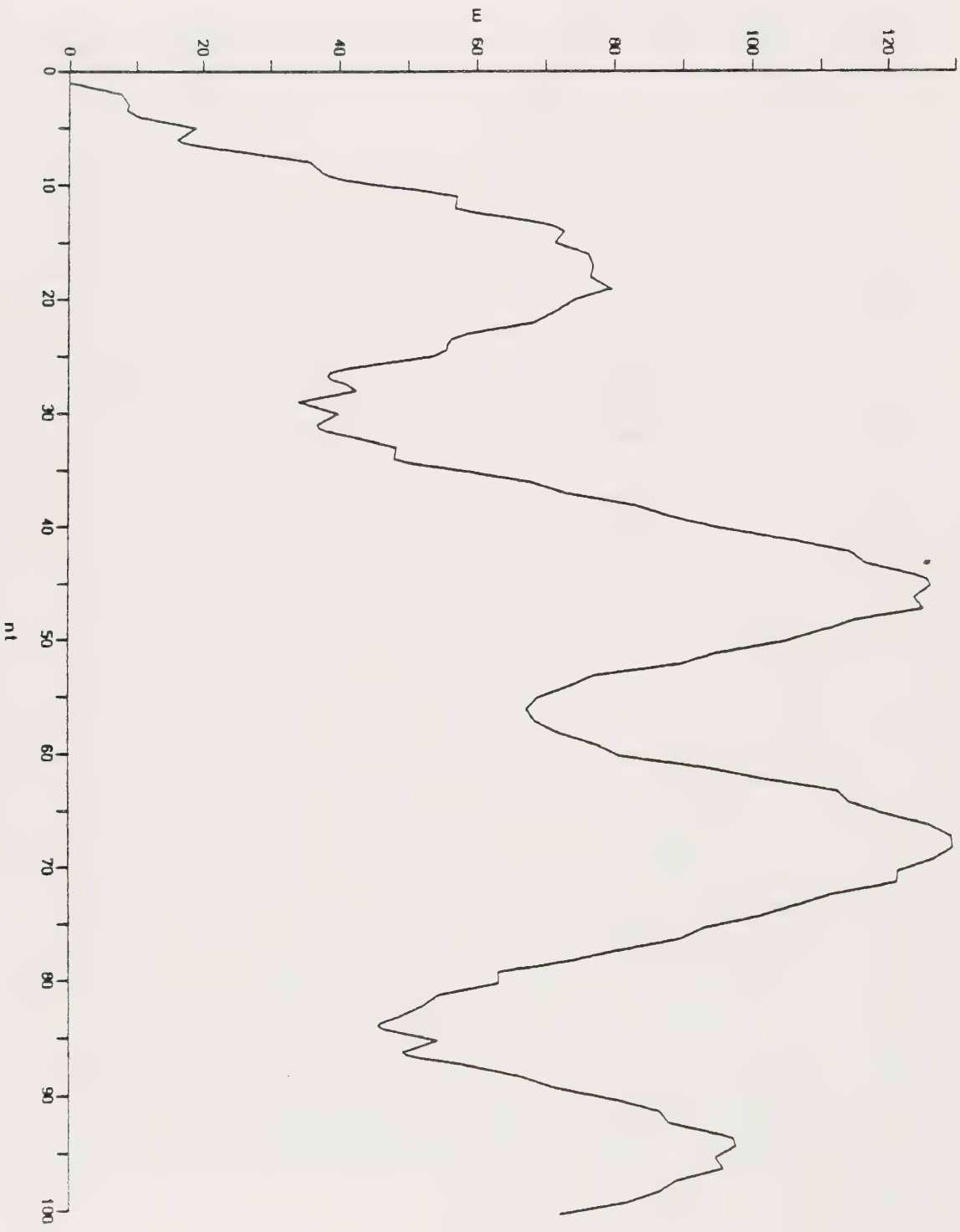

Figure 2

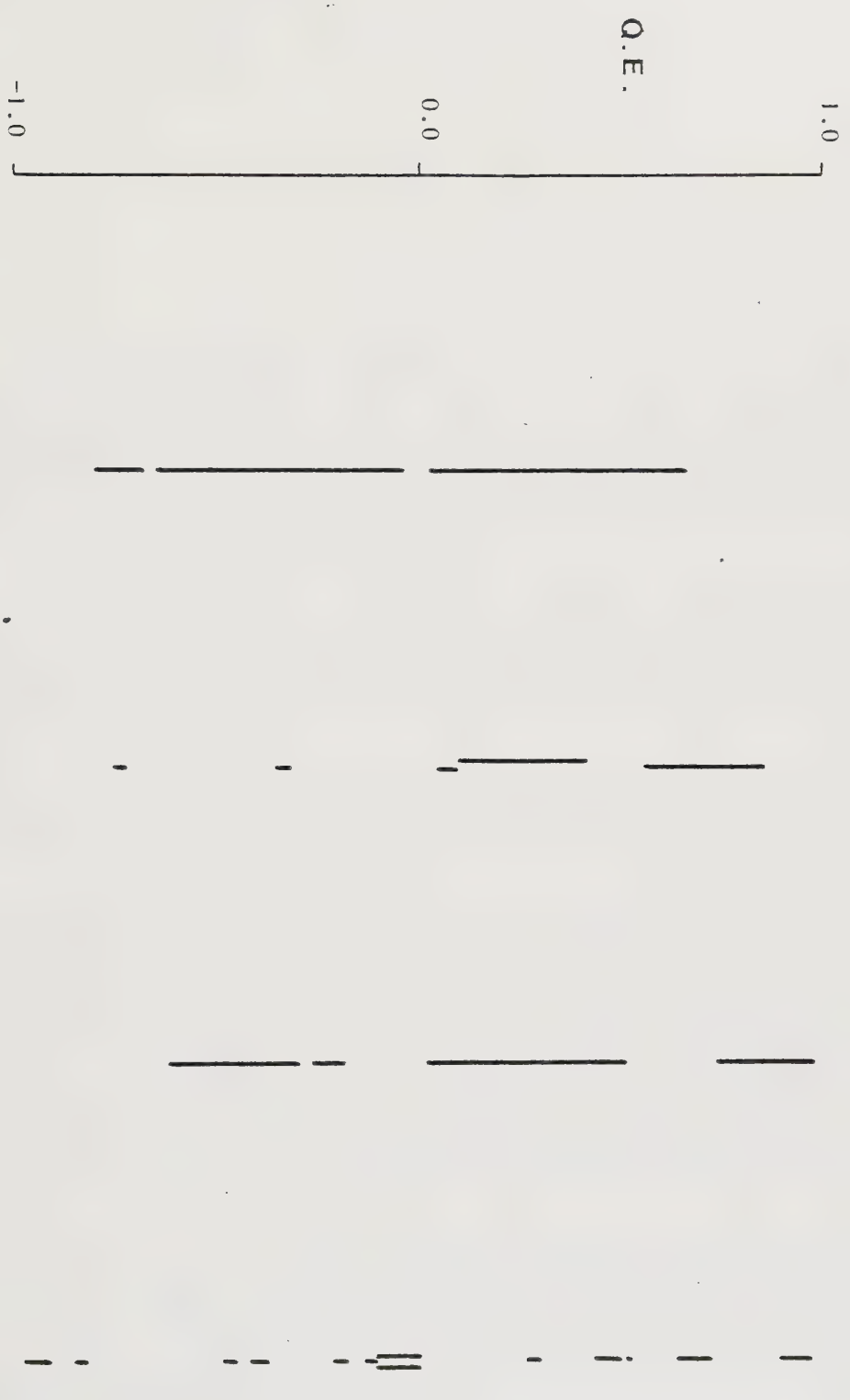

Figure 9

CONCLUSION

Dans cette thèse, nous nous sommes proposé l'étude de l'intégrabilité des systèmes dynamiques, en considérant surtout des systèmes avec un nombre de degrés de liberté fini, régis par des équations différentielles ordinaires. Pour de tels systèmes, la recherche directe de l'intégrabilité, basée sur le calcul explicite des constantes du mouvement n'est pas simple, c'est pourquoi nous avons utilisé des méthodes annexes pour détecter les systèmes intégrables.

En particulier, nous avons implémenté la méthode de Painlevé dans ce cadre précis. Cette méthode, proposée à l'origine par Ablowitz, Ramani et Ségur pour l'étude des P.D.E., permet aussi de sélectionner des candidats d'intégrabilité dans le cadre des O.D.E., aussi bien pour des systèmes hamiltoniens que dissipatifs.

Au fur et à mesure que nous avons progressé dans la découverte de nouveaux systèmes intégrables, le besoin s'est fait sentir d'une meilleure compréhension de la propriété de Painlevé en elle-même.

En particulier, nous avons été amenés à affiner la conjecture de manière à rendre compte de certains systèmes hamiltoniens intégrables qui ne possèdent pas la propriété de Painlevé originelle. Ceci nous a conduit à définir une propriété de Painlevé faible pour les systèmes hamiltoniens à deux dimensions. Pour les systèmes dissipatifs, il apparaît même quelquefois que l'intégrabilité puisse exister indépendamment de la propriété de Painlevé. Ces résultats renforcent encore le caractère *essentiellement prédictif* de l'analyse des singularités : c'est un moyen très utile et très direct pour identifier les systèmes intégrables qui sans cela seraient très difficiles à découvrir, ceci sans prétendre donner une description exhaustive de tous les cas intégrables.

Cette étude nous a aussi permis de mieux préciser la notion d'intégrabilité pour un système dynamique. Nous avons été conduit à admettre pour intégrables non seulement les équations qui sont solubles par quadratures mais aussi les équations qui, d'une certaine manière, sont linéarisables, qu'elles se ramènent soit à une équation différentielle ordinaire linéaire, soit à des transcendantes de Painlevé, qui sont linéarisables au moyen d'une équation intégrale.

A ce jour, aucun exemple de transcendantes d'ordre strictement plus grand que deux n'est encore connu, mais on connaît déjà de bons candidats (voir chapitre 6).

Pour terminer, on ne peut que souligner le fait que cette thèse est loin d'avoir épuisé ce domaine en pleine expansion. L'application des critères de Painlevé ("fort et faible") devrait pouvoir donner des résultats en dimensions supérieures (trois ou plus), et s'est déjà montrée fructueuse pour les systèmes avec un nombre infini de degré de liberté, ainsi que certains de nos résultats récents permettent de le supposer.

DORIZZI Bernadette

ETUDE DE L'INTEGRABILITE DES SYSTEMES DYNAMIQUES

RESUME :

Cette thèse traite de l'intégrabilité de systèmes dynamiques modélisés par des équations différentielles ordinaires, non linéaires.

L'analyse des singularités des solutions des équations du mouvement fournit un critère très utile pour sélectionner des systèmes potentiellement intégrables. (Propriété de Painlevé). Plusieurs nouveaux systèmes intégrables, tant Hamiltoniens que dissipatifs sont ainsi exhibés en lien avec cette propriété.

Cette étude a conduit à affaiblir la propriété de Painlevé originelle, de manière à rendre compte d'un plus grand nombre de systèmes intégrables.

Enfin, les mêmes questions de chaos et d'intégrabilité sont posées pour les systèmes quantiques : En particuliers, ont été étudiés la quantification de systèmes classiquement intégrables ainsi qu'un modèle quantique simple dont l'analogue classique est connu pour son comportement chaotique.

MOTS CLEFS :

Systèmes dynamiques, non linéaire, intégrabilité, Painlevé (propriété de) quantification, chaos quantique, Hamiltoniens (systèmes), Dissipatifs (systèmes).